

QUANTUM KIRWAN MORPHISM AND GROMOV-WITTEN INVARIANTS OF QUOTIENTS

C. WOODWARD

ABSTRACT. We construct a quantum version of the Kirwan map from the equivariant quantum cohomology $QH_G(X)$ of a smooth complex projective variety X with the action of a connected complex reductive group G to the orbifold quantum cohomology $QH(X//G)$ of its geometric invariant theory quotient $X//G$, and prove that it intertwines the genus zero gauged Gromov-Witten potential of X with the genus zero Gromov-Witten graph potential of $X//G$. Finally we give a formula for a solution to the quantum differential equation on $X//G$, in terms of a localized gauged potential for X .

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1. INTRODUCTION

This paper constructs a quantum version of the map studied by Kirwan [46] from the equivariant cohomology to the cohomology of the geometric invariant theory (git) quotient, as suggested by Salamon and Ziltener [98], [99], [100], under the condition that the target is a smooth projectively-embedded variety with a connected reductive group action such that the stable locus is equal to the semistable locus. We then show that the quantum Kirwan map intertwines the Gromov-Witten graph potential of the quotient with the *gauged Gromov-Witten* potential of the action in the large area limit. In the physics language, the quantum Kirwan map relates correlators of a (possibly non-linear, non-abelian) gauged sigma model with those of the sigma model of the git quotient. As such, the results overlap with those of Givental [31], Lian-Liu-Yau [55], Iritani [39] and others. The connection to mirror symmetry is explained in the paper of Hori-Vafa [38]: because mirror symmetry for vector spaces is rather trivial, the non-trivial change of coordinates arises when passing from a gauged linear sigma model to the

sigma model for the quotient. Since the quantum Kirwan map is defined geometrically, it can be rather difficult to compute and the algebraic approach in [31], [55] is more effective in cases where it applies. However the geometric approach pursued here has the advantage over the approach in [31], [55] that there are no semipositivity assumptions on $X//G$ or abelian-ness assumptions on the group G . Also, X can be a projective variety rather than a vector space.

To explain the gauged Gromov-Witten potential, recall that the *equivariant cohomology* of a G -space X is the cohomology of the homotopy quotient $X_G = EG \times_G X$ where $EG \rightarrow BG$ is a universal G -bundle. Equivariant quantum cohomology should count maps $u : C \rightarrow X_G$ where C is a curve equipped with some additional data and u is a holomorphic map. Any such map can be viewed as a map to $BG = EG/G$ together with a lift to X_G . Holomorphic maps to BG correspond to holomorphic G -bundles, and so a holomorphic map to X_G is given by a holomorphic G -bundle $P \rightarrow C$ together with a holomorphic section of the associated X bundle $u : C \rightarrow P(X) := P \times_G X$. Givental [31] had earlier introduced an *equivariant Gromov-Witten theory*, based on equivariant counts of maps from a curve to X . These counts give rise to a family of products on the *equivariant quantum cohomology* $QH_G(X) = H_G(X) \otimes \Lambda_X^G$ where convergence issues are solved by the introduction of the *Novikov field* $\Lambda_X^G \subset \text{Hom}(H_2^G(X, \mathbb{Z}), \mathbb{Q})$. In the language of maps to the classifying space X_G , Givental's equivariant Gromov-Witten theory corresponds to counting maps from C to X_G whose image lies in a single fiber of the projection $X_G \rightarrow BG$, that is, such that the G -bundle is trivial. In order to distinguish this theory from Givental's, we will call the theory with non-trivial bundles *gauged Gromov-Witten theory*, and call the map $u : C \rightarrow X_G$ a *gauged map* to X .

Gauged Gromov-Witten invariants should be defined as integrals over moduli spaces of gauged maps. In order to obtain proper moduli spaces one needs to impose a *stability* or *moment map* condition as well as compactify the moduli space by, for example, allowing *bubbling*. Mundet and Salamon's work, see [16], provides a symplectic approach to the moduli spaces of gauged maps and construction of invariants in the case that the target is a vector space. Mundet's thesis [66] connects the symplectic approach to an algebraic stability condition, which combines the Ramanan stability condition for principal bundles with the Hilbert-Mumford stability for the action. Schmitt [83], [84] had earlier constructed a Grothendieck-style compactification in the case that the target is a smooth projective variety, while in the symplectic setting Mundet [67] and Ott [77] show that the connected components of the moduli spaces of semistable gauged maps have a Kontsevich-style compactification.

In general, one needs virtual fundamental cycles to define integration over the moduli spaces. Here we restrict to the case that the target X is a projective G -variety and note that a pair $(P \rightarrow C, u : C \rightarrow P(X))$ is by definition a morphism from C to the *quotient stack* X/G introduced by Deligne-Mumford [21]. We then use the theory of virtual fundamental classes developed by Behrend-Fantechi [9], based on earlier work of Li-Tian [54], to define gauged Gromov-Witten invariants. The symplectic geometry is then only used as motivation, and to show that the Deligne-Mumford stacks that arise are proper. Of course it is desirable to have semistable reduction theorems to show properness but from the algebraic perspective even the stability conditions are somewhat obscure and we prefer the symplectic route to properness. A good example is the moduli space of semistable bundles on a curve, where properness is immediate from the Narasimhan-Seshadri description as unitary representations of the fundamental group but semistable reduction is somewhat tricky.

The theory of gauged Gromov-Witten invariants in genus zero fits into an algebraic formalism that is a “complexification” of the theory of *homotopy associative*, or A_∞ , algebras introduced by Stasheff [86]. In the A_∞ story, which roughly corresponds to “open strings” in the mathematical physics language, one has notions of A_∞ algebras, A_∞ morphisms, and A_∞ traces. These notions are associated with different polytopes called the associahedra, multiplihedra, and cyclohedra respectively. The complexifications of these spaces are the Grothendieck-Knudsen space $\overline{\mathcal{M}}_{0,n}$ of stable n -marked genus 0 curves, Ziltener compactification $\overline{\mathcal{M}}_{n,1}(\mathbb{A})$ of n -marked 1-scaled affine lines, and the Fulton-MacPherson space $\overline{\mathcal{M}}_n(\mathbb{P}) := \overline{\mathcal{M}}_{0,n}(\mathbb{P}, [\mathbb{P}])$ of stable n -marked, genus 0 maps of class $[\mathbb{P}]$ to the projective line \mathbb{P} respectively. The first space leads to the notion of *genus zero cohomological field theory (CohFT)*, in particular, a *CohFT algebra* given by the invariants with multiple incoming markings and a single outgoing marking. The second space is associated to the notion of *morphism of CohFT algebras*, and the third to the notion of *trace on a CohFT algebra*. The following is proved in Gonzalez-Woodward [33], and is reviewed in Theorem 7.20 below.

Theorem 1.1 (Gauged Gromov-Witten invariants). [33] *Let X be a smooth polarized projective G -variety and C a smooth connected projective curve. Suppose that every semistable gauged map is stable; then the category of stable gauged maps is a proper Deligne-Mumford stack equipped with a perfect obstruction theory. The gauged Gromov-Witten invariants $(\tau_X^{G,n})_{n \geq 0} : QH_G(X)^n \times H_n(\overline{\mathcal{M}}_n(\mathbb{P})) \rightarrow \Lambda_X^G$ define a trace on $QH_G(X)$.*

The gauged Gromov-Witten invariants are also defined for polarized quasiprojective varieties under suitable properness assumptions for the moduli stacks of gauged maps, for example, if the polarization corresponds to an equivariant symplectic form with proper moment map convex at infinity. In particular, gauged Gromov-Witten invariants are defined for a vector space X equipped with the action of a torus G such that the weights are contained in an open half-space in the space of rational weights; this condition implies in particular that the quotient $X//G$ is proper under the stable=semistable condition.

One can organize the gauged Gromov-Witten invariants into a formal *gauged Gromov-Witten potential*

$$\tau_X^G : QH_G(X) \rightarrow \mathbb{Q}, \quad \alpha \mapsto \sum_{n \geq 0} \tau_X^{G,n}(\alpha, \dots, \alpha; 1)/n!.$$

The splitting axiom implies that the bilinear form constructed from the second derivatives of the potential is compatible with the quantum product \star_α on $T_\alpha QH_G(X)$ in the usual sense. However this bilinear form will usually be degenerate and so will not define a family of Frobenius algebra structures. It is convenient to throw into the definition of the trace certain *Liouville classes* on the moduli space of gauged maps, in which case, if G is trivial, τ_X^G becomes equivalent to the *graph potential* considered in Givental [31].

From the definition of Mundet stability one expects the gauged Gromov-Witten invariants to be related in the limit that the equivariant symplectic class $[\omega_{X,G}]$ approaches infinity to the Gromov-Witten invariants of the quotient $X//G$, or more precisely, to the genus zero *graph potential* for the geometric invariant theory quotient $X//G$

$$\tau_{X//G} : QH(X//G) \rightarrow \Lambda_{X//G}.$$

For our purposes, it is more natural to work over the larger Novikov field Λ_X^G . From the symplectic point of view the study of the large area limit of the vortex equations was initiated

by Gaio-Salamon [29] and Ziltener [98], who suggested a map $\kappa_X^G : QH_G(X) \rightarrow QH(X//G)$ by counting “affine vortices”. By a Hitchin-Kobayashi correspondence with Venugopalan [92], [93] these maps correspond to the following algebraic objects:

Definition 1.2. (Affine gauged maps) In the case that $X//G$ is a free quotient, an *affine gauged map* to X consists of a tuple

$$(p : P \rightarrow \mathbb{P}, \lambda : \mathbb{P} \rightarrow T^\vee \mathbb{P} \otimes \mathcal{O}(2\infty)), u : \mathbb{P} \rightarrow P(X), \underline{z} \in (\mathbb{P})^n$$

consisting of

- (a) (Scaling form) a meromorphic one-form λ on \mathbb{P} with only a double pole at $\infty \in \mathbb{P}$ (hence inducing an affine structure on $\mathbb{P} - \{\infty\}$);
- (b) (Morphism to the quotient stack, stable at infinity) a morphism $\mathbb{P} \rightarrow X/G$, consisting of a G -bundle $p : P \rightarrow \mathbb{P}$ and a section $u : \mathbb{P} \rightarrow P(X)$ such that $u(\infty)$ is contained in the open subvariety $P(X^{\text{ss}})$ associated with the semistable locus $X^{\text{ss}} \subset X$; and
- (c) (Markings) an n -tuple of distinct points $\underline{z} = (z_1, \dots, z_n) \in (\mathbb{P})^n$.

In the case that the action of G on the semistable locus of X is only locally free we also allow a stacky structure at infinity. That is, for $r > 0$ an integer we denote by μ_r the group of r -th complex roots of unity and by

$$\mathbb{P}[1, r] := (\mathbb{C}^2 - \{0\})/\mathbb{C}^\times$$

the weighted projective line with a μ_r -singularity at ∞ , where \mathbb{C}^\times acts on \mathbb{C}^2 with weights $1, r$. The morphism u is then required to be a representable morphism from some $\mathbb{P}[1, r]$ to X/G .

Theorem 1.3 (Quantum Kirwan Morphism). *Suppose that X is a projective G -variety equipped with a polarization such that every semistable point is stable. Integrating over a compactified stack of affine gauged maps defines a morphism of CohFT algebras*

$$\kappa_X^G : QH_G(X) \rightarrow QH(X//G).$$

By definition a morphism of CohFT algebras consists of a sequence of maps

$$\kappa_X^{G,n} : QH_G(X)^n \times H(\overline{\mathcal{M}}_{n,1}(\mathbb{A})) \rightarrow QH(X//G), \quad n \geq 0$$

satisfying a splitting axiom that guarantees that the formal map

$$\kappa_X^G : QH_G(X) \rightarrow QH(X//G), \quad \alpha \mapsto \sum_{n \geq 0} \kappa_X^{G,n}(\alpha, \dots, \alpha; 1)/n!$$

induces a \star -homomorphism on each tangent space. In the case that the *curvature* $\kappa_X^{G,0}$ of the quantum Kirwan morphism vanishes, one obtains in particular a morphism of small quantum cohomology rings

$$\kappa_X^{G,1} : T_0 QH_G(X) \rightarrow T_0 QH(X//G).$$

In this sense, morphisms of CohFT algebras can be considered as non-linear generalizations of algebra homomorphisms.

More precisely the quantum Kirwan morphism is defined by virtual integration over a compactification $\overline{\mathcal{M}}_{n,1}^G(\mathbb{A}, X)$ of the moduli stack of affine gauged maps equipped with evaluation and forgetful maps

$$\text{ev} \times \text{ev}_\infty : \overline{\mathcal{M}}_{n,1}^G(\mathbb{A}, X) \rightarrow (X/G)^n \times \overline{I}_{X//G}, \quad f : \overline{\mathcal{M}}_{n,1}^G(\mathbb{A}, X) \rightarrow \overline{\mathcal{M}}_{n,1}(\mathbb{A})$$

where $\bar{I}_{X//G}$ is the rigidified inertia stack appearing in orbifold Gromov-Witten theory. Properness of this moduli space follows from compactness results of Ziltener [98], [100] (for affine vortices) and a Hitchin-Kobayashi correspondence for affine vortices due to Venugopalan and the author, to appear. This space again has a perfect relative obstruction theory, and pull-push using the virtual fundamental class gives rise to the maps $\kappa_{X,n}^G$. From the physics point of view, Witten [97] explained the relationship between correlators in gauged sigma models and sigma models with target the symplectic quotient as a kind of “renormalization” given by “counting pointlike instantons”. The “quantum Kirwan map” is a precise mathematical meaning for this statement for arbitrary gauged (possibly non-linear) sigma models.

Despite the complicated-looking definition, the stack of affine gauged maps is easy to understand in simple cases.

Example 1.4. (Toric orbifolds) Let X a vector space and G a torus with Lie algebra \mathfrak{g} acting with weights $\mu_1, \dots, \mu_k \in \mathfrak{g}^\vee$ contained in a half-space a real form $\mathfrak{g}_{\mathbb{R}}^\vee$. Assume that stable=semistable, so that the quotient $X//G$ is a proper toric Deligne-Mumford stack. Then $\mathcal{M}_{1,1}^G(\mathbb{A}, X)$ is isomorphic to the stack of morphisms u from \mathbb{A} to X (that is, X -valued polynomials in a single variable) satisfying the following conditions for any $d \in \mathfrak{g}_{\mathbb{Q}}$:

- (a) (Degree Restriction) the j -th component u_j of u has degree at most (d, μ_j) , $j = 1, \dots, k$, that is,

$$u_j(z) = a_{j,0} + a_{j,1}z + \dots + a_{j, \lfloor (d, \mu_j) \rfloor} z^{\lfloor (d, \mu_j) \rfloor}$$

for some constants $a_{j,k} \in \mathbb{C}$. If (d, μ_j) is not an integer let $a_{j, (d, \mu_j)} = 0$.

- (b) (Stability condition) The collection of leading order coefficients

$$u(\infty) := (a_{j, (d, \mu_j)})_{j=1}^k \in X^{\exp(d)}$$

lies in the semistable locus X^{ss} of X and so defines a point in the substack of the inertia stack given by $\{\exp(d) \times X^{\exp(d), \text{ss}}\}/G \subset I_{X//G}$.

If $X//G$ is Fano with minimal Chern number at least two then $\kappa_X^G(0) = 0$ and $D_0 \kappa_X^G$ is an algebra homomorphism of small quantum cohomologies. Thus knowing the map $D_0 \kappa_X^G$ allows to give a presentation of the small quantum cohomology of $X//G$.

- (a) (Projective space) If $G = \mathbb{C}^\times$ acts by scalar multiplication on $X = \mathbb{C}^k$ then there is a unique morphism $u : \mathbb{A} \rightarrow X$ such that all components have degree 1,

$$u(z) = (a_{1,0} + a_{1,1}z, \dots, a_{k,0} + a_{k,1}z)$$

with marking at $z_1 = 0$, given limit at infinity

$$\text{ev}_\infty u = [a_{1,1}, \dots, a_{k,1}] \in \mathbb{P}^{k-1} = X//G$$

and vanishing value

$$\text{ev}_0 u = u(0) = (a_{1,0}, \dots, a_{k,0}) \in \mathbb{C}^k = X$$

at $z_1 = 0 \in \mathbb{C} \cong \mathbb{A}$. This implies that

$$D_0 \kappa_X^G(\xi^k) = q$$

where ξ is the generator of $QH_G(X)$ and $k = \dim(X)$. Hence the presentation

$$QH(X//G) = \Lambda_X^G[\xi]/(\xi^k - q)$$

of quantum cohomology of projective space.

- (b) (The teardrop orbifold, a weighted projective line) If $G = \mathbb{C}^\times$ acts on $X = \mathbb{C}^2$ with weights 1, 2, so that $X//G = \mathbb{P}[1, 2]$ is the teardrop orbifold, then $\overline{M}_{1,1}^G(\mathbb{A}, X, 0) = X//G$ implies that $D_0\kappa_X^G(1) = 1$ and $D_0\kappa_X^G(\xi)$ is the point class. To evaluate $D_0\kappa_X^G((2\xi)^{2d}\xi^d)$ interpret $(2\xi)^{2d}\xi^d$ as the Euler class of the vector bundles corresponding to the derivatives of u at 0. Morphisms of class $d \in \mathbb{Q} \cong H_2^G(X, \mathbb{Q})$ are given by pairs of polynomials

$$u(z) = (a_{1,0} + \dots + a_{1,[d]}z^{[d]}, a_{1,1} + \dots + a_{1,2d}z^{2d}).$$

Integrating the Euler class counts such maps whose derivatives $j!a_{1,j}, j < d, j!a_{2,j}, j < 2d$ zero at the marking and, if d is even, semistable leading order terms

$$\text{ev}_\infty u = [a_{1,d}, a_{2,2d}] \in \mathbb{P}[1, 2].$$

are fixed to be a given point. There is a unique such map. For $d = 1/2$ this implies

$$D_0\kappa_X^G(\xi^2) = 1_{\mathbb{Z}_2}/2$$

half the generator $1_{\mathbb{Z}_2}$ of the twisted sector. For $d = 1$ this implies that

$$D_0\kappa_X^G(\xi^3) = q/4.$$

Hence the presentation of the quantum cohomology of the teardrop orbifold

$$QH(\mathbb{P}[1, 2]) = \Lambda_X^G[\xi]/(q - 4\xi^3)$$

which is a special case of Coates-Lee-Corti-Tseng [18].

See Example 5.32 for more details. A similar strategy gives relations for the quantum cohomology of any toric orbifold. In addition, the setup also applies to quotients by non-abelian groups such as quiver varieties.

In the large area limit the graph potentials $\tau_X^G, \tau_{X//G}$ are naturally related via the quantum Kirwan morphism: We fix a symplectic class $[\omega_{X,G}] \in H_G^2(X)$ and consider the stability conditions corresponding to the classes $\rho[\omega_{X,G}]$ as $\rho \in (0, \infty)$ varies.

Theorem 1.5 (Adiabatic Limit Theorem). *Suppose that C is a smooth projective curve and X a polarized projective G -variety such that stable=semistable for gauged maps from C to X of sufficiently large ρ . Then*

$$(1) \quad \tau_{X//G} \circ \kappa_X^G = \lim_{\rho \rightarrow \infty} \tau_X^G.$$

In other words, the diagram

$$\begin{array}{ccc} QH_G(X) & \xrightarrow{\kappa_X^G} & QH(X//G) \\ \searrow \tau_X^G & & \swarrow \tau_{X//G} \\ & \Lambda_X^G & \end{array}$$

commutes in the limit $\rho \rightarrow \infty$. The equality, or commutativity of the diagram, holds in the space of distributions in q , in other words, for each power of q separately. Mathematicians familiar with equivariant symplectic geometry in the last millennium will recognize the similarity with “quantization commutes with reduction” theorems, see for example Guillemin-Sternberg

[37]. However, in the intervening years the use of “quantum” is mostly changed so now it refers to holomorphic curves. The terminology “adiabatic” arises from the fact that stable gauged maps correspond to minima of an energy function depending on ρ , so the theorem relates minima in the limit $\rho \rightarrow \infty$.

One is often interested not in the graph Gromov-Witten invariants but rather in the *localized graph potential* that is a solution $\tau_{X//G,-}$ to the quantum differential equation for $X//G$: for $\alpha \in QH(X//G)$, $\xi \in T_\alpha QH(X//G)$

$$\tau_{X//G,-} : QH(X//G) \rightarrow QH(X//G)[[\hbar^{-1}]], \quad \hbar \partial_\nu \tau_{X//G,-}(\alpha) = \nu \star_\alpha \tau_{X//G,-}(\alpha).$$

The components of $\tau_{X//G,-}$ satisfy a version of the Picard-Fuchs equations which play an important role in mirror symmetry [31]. The function $\tau_{X//G,-}$ can be expressed as the “contribution from 0” in the fixed point formula for the graph potential for \mathbb{P} , induced by the circle action on S^1 . (There is also a contribution $\tau_{X//G,+}$ from the fixed point $\infty \in \mathbb{P}$.) There are similar gauged versions

$$\tau_{X,\pm}^G : QH_G(X) \rightarrow QH_G(X)[[\hbar^{-1}]]$$

which capture the contributions from $0, \infty \in \mathbb{P}$ to the localization formula applied to the *gauged* graph potential. Similarly the quantum Kirwan map has an S^1 -equivariant extension $\kappa_{X,G} : QH_G(X) \rightarrow QH(X//G)[[\hbar]]$, with target formal power series with quotient modulo any power q^d a polynomial in \hbar . The factorization of the graph potentials generalizes to the gauged setting: $\tau_X^G = (\tau_{X,-}^G, \tau_{X,+}^G)$ and we prove a localized adiabatic limit Theorem 1.6 below for the contributions to the fixed point formula. Let $\kappa_X^{G,\text{class}}$ denote the classical Kirwan map extended to $QH(X//G)$ by linearity over Λ_X^G .

Theorem 1.6 (Localized adiabatic limit theorem). *In the setting of Theorem 1,*

$$\tau_{X//G,-} \circ \kappa_X^G = \lim_{\rho \rightarrow \infty} \kappa_X^{G,\text{class}} \circ \tau_{X,-}^G.$$

In other words, after composition with the quantum Kirwan map the localized graph potential is given by the classical Kirwan map applied to the localized gauged potential. The result also holds in the “twisted case”, that is, after inserting the Euler class of the index of an equivariant bundle on the target, which in good cases describes the localized graph potentials of complete intersections. In this way one obtains a generalization of the “mirror formulas” of Givental [31], Lian-Liu-Yau [55], Iritani [39] and others for localized graph potentials [31] in the toric case to arbitrary geometric invariant theory quotients. However, the approach here is different from that of [31], [55] etc. in that the “mirror map” is expressed as integrals over moduli spaces, while the approach of [31], [55] etc. solves for the “mirror map” as the solution to an algebraic equation.

The generality of these results allows various applications, mostly developed jointly with E. Gonzalez, which will appear elsewhere. We were rather surprised to discover that many of the “standard formulas” from classical equivariant symplectic geometry generalize to the quantum case by substituting the quantum Kirwan morphism for the classical Kirwan map; this is rather unexpected since all of these formulas involve functoriality of cohomology in some way which is generally lacking in the quantum setting. For example, we claim that there is (i) a wall-crossing “quantum Kalkman” formula for Gromov-Witten invariants under variation of git, including invariance in the case of crepant flops (ii) an abelianization “quantum Martin”

formula relating Gromov-Witten invariants of quotients by connected reductive groups and their maximal tori, first suggested by Hori-Vafa [38, Appendix] and Bertram-Ciocan-Fontanine-Kim [11] and (iii) a quantum version of Witten's non-abelian localization principle, relating the equivariant quantum cohomology correlators for X with the quantum cohomology correlators for $X//G$.

2. TRACES AND MORPHISMS OF COHOMOLOGICAL FIELD THEORY ALGEBRAS

To state the main result precisely, we have to explain what it means to have a “commutative diagram” of cohomological field theories. In this section we describe the moduli spaces of stable curves (complexified associahedron in genus zero), stable parametrized curves (complexified cyclohedron) and stable affine scaled curves (complexified multiplihedron), which lead to the notion of CohFT algebra, trace on a CohFT algebra, and morphism of CohFT algebras respectively, in analogy with the theory of A_∞ spaces, morphisms, and traces. Then we introduce notions of compositions of morphisms and traces, or morphisms of CohFT algebras, which are analogous to the composition of the corresponding A_∞ notions. This makes CohFT algebras into a kind of ∞ -category. Notably, we do not have a version of complexified multiplihedron for higher genus curves, which is why the theory here is restricted to genus zero.

2.1. Complexified associahedron and CohFT algebras. Moduli spaces of stable curves were introduced by Mayer and Mumford in 1964 [65], and further studied in Deligne-Mumford [21]. Moduli of stable marked curves were studied by Grothendieck in 1968 and later by Knudsen [47]. In this section we describe these compactifications and the notion of *cohomological field theory* introduced by Kontsevich-Manin, see [57]. We remark that since the notion of stack is not introduced until Section 4, we avoid it until then and adopt the point of view that the moduli spaces are just topological spaces. First we describe stable curves.

Definition 2.1. Let $n \geq 0$ be an integer.

- (a) (Nodal curves) An n -marked nodal curve consists of a projective nodal curve C with an n -tuple of distinct, smooth marked points $\underline{z} = (z_1, \dots, z_n) \in C^n$. An *isomorphism* of n -marked nodal curves $(C, \underline{z}), (C', \underline{z}')$ is an isomorphism $\phi : C \rightarrow C'$ such that $\phi(z_i) = z'_i$ for $i = 1, \dots, n$.
- (b) (Stable curves) A nodal n -marked curve $C = (C, \underline{z})$ is *stable* iff C has finite automorphism group. That is, each genus zero component has at least three *special points* (nodes or markings) and each genus one component has at least one special point. Note that we do not require C to be connected.
- (c) (Modular graphs) The combinatorial type Γ of a stable curve is a *modular graph*:
 - (i) (Graph) an unoriented graph $\Gamma = (\text{Vert}(\Gamma), \text{Edge}(\Gamma))$ whose vertices correspond to irreducible components of C , finite edges $\text{Edge}_{<\infty}(\Gamma)$ to nodes, and semi-infinite edges $\text{Edge}_\infty(\Gamma)$ to markings, equipped with a
 - (ii) (Genus function) $\mathbb{Z}_{\geq 0}$ -valued function $g : \text{Vert}(\Gamma) \rightarrow \mathbb{Z}_{\geq 0}$ recording the genus of each irreducible component of C and
 - (iii) (Labelling of semi-infinite edges) a bijection $l : \text{Edge}_\infty(\Gamma) \rightarrow \{1, \dots, n\}$ of the semi-infinite edges with labels $1, \dots, n$.

A modular graph is *stable* if it corresponds to a stable curve. That is, each vertex with label 0 resp. 1 has valence at least 3 resp 1.

Any stable curve C has a *universal deformation* given by a family of curves $\pi : C_S \rightarrow S$ over a parameter space S uniquely defined up to isomorphism, see for example [5]. Let $\overline{M}_{g,n}$ denote the set of isomorphism classes of connected genus g , n -marked stable curves. More naturally one should consider the moduli *stack* of stable curves but we put off discussion of stacks to Section 4. In the case of genus zero curves the universal deformation, and topology on $\overline{M}_{g,n}$, have a simple description [5, p. 176], [62, Appendix D]: For any marking z_i and irreducible component C_j of C we denote by z_i^j the node in C_j connecting to the irreducible component of C containing z_i , or z_i if z_i is contained in C_j .

Definition 2.2. (Convergence of a sequence of stable curves) A sequence $[C_\nu]$ converges to $[C]$ in $\overline{M}_{g,n}$ if C_ν is isomorphic to $\pi^{-1}(s_\nu)$ for a sequence s_ν converging to s in the base S of the universal deformation. Explicitly, if $g = 0$, a sequence $[(C_\nu, z_{1,\nu}, \dots, z_{n,\nu})]$ with smooth domain C_ν converges to $[(C, z_1, \dots, z_n)]$ if there exists, for each irreducible component C_j of C , a sequence of holomorphic isomorphisms $\phi_{j,\nu} : C_j \rightarrow C_\nu$ such that

- (a) (Limit of a marking) for all i, j , $\lim_{\nu \rightarrow \infty} \phi_{j,\nu}^{-1}(z_{i,\nu}) = z_i^j$; and
- (b) (Limit of a different parametrization) for all $j \neq k$, $\lim_{\nu \rightarrow \infty} \phi_{j,\nu} \phi_{k,\nu}^{-1}$ has limit the constant map with value the node of C_j connecting to C_k .

With the topology induced by this notion of convergence, $\overline{M}_{g,n}$ is compact and Hausdorff (and in fact, a projective variety [21].) For any possible disconnected graph Γ we denote by $M_{g,n,\Gamma}$ the space of isomorphism classes of curves of combinatorial type Γ with n semiinfinite edges and total genus g , and $\overline{M}_{g,n,\Gamma}$ its closure. If $\Gamma = \Gamma_0 \sqcup \Gamma_1$ is a disjoint union then $\overline{M}_{g,n,\Gamma} = \overline{M}_{g,n,\Gamma_0} \times \overline{M}_{g_1,n_1,\Gamma_1}$. The moduli spaces of stable marked curves $\overline{M}_{n,\Gamma}$ satisfy a natural functoriality with respect to morphisms of modular graphs Γ .

Definition 2.3. (Morphisms of modular graphs) A *morphism* of modular graphs $\Upsilon : \Gamma \rightarrow \Gamma'$ is a surjective morphism of the set of vertices $\text{Vert}(\Gamma) \rightarrow \text{Vert}(\Gamma')$ obtained by combining the following: (these are called *extended isogenies* in Behrend-Manin [10])

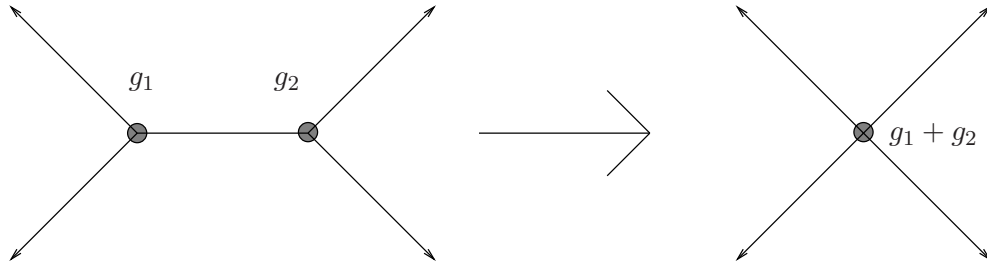


FIGURE 1. Collapsing an edge

- (a) Υ *collapses an edge* if the map on vertices $\text{Vert}(\Upsilon) : \text{Vert}(\Gamma) \rightarrow \text{Vert}(\Gamma')$ is a bijection except for a single vertex $v' \in \text{Vert}(\Gamma')$ which has two pre-images connected by an edge in $\text{Edge}(\Gamma)$, and $\text{Edge}(\Gamma') \cong \text{Edge}(\Gamma) - \{e\}$. The genus function on Γ' is obtained by push-forward that is, $g(v') = \sum_{\Upsilon(v)=v'} g(v)$.



FIGURE 2. Collapsing a loop

- (b) Υ *collapses a loop* if the map on vertices is a bijection and $\text{Edge}(\Gamma') \cong \text{Edge}(\Gamma) - \{e\}$ where e is an edge connecting a vertex v to itself. Then the genus functions on Γ, Γ' are identical except at v where $g'(v) = g(v) + 1$.
- (c) Υ *cuts an edge* $e \in \text{Edge}(\Gamma)$ if the map on vertices is a bijection, but $\text{Edge}(\Gamma') \cong \text{Edge}(\Gamma) - \{e\} + \{e_+, e_-\}$ where e_{\pm} are semiinfinite edges attached to the vertices contained in e . Any morphism $\Upsilon : \Gamma \rightarrow \Gamma'$ cutting an edge induces an isomorphism $\overline{M}(\Upsilon) : \overline{M}_{g,n,\Gamma'} \rightarrow \overline{M}_{g,n,\Gamma}$ obtained by identifying the markings corresponding to the edges e_{\pm} in Γ' .

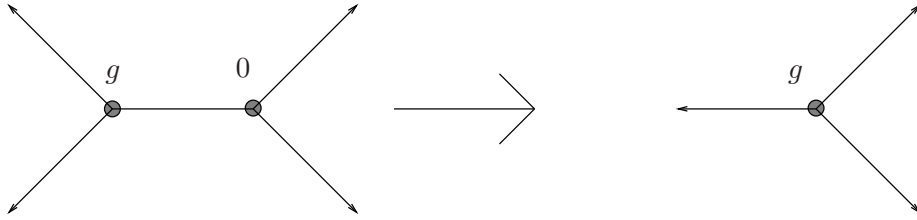


FIGURE 3. Forgetting a tail

- (d) Υ *forgets a tail* (semiinfinite edge) $e \in \text{Edge}^{\infty}(\Gamma)$ if the map on vertices is a bijection, but $\text{Edge}(\Gamma') \cong \text{Edge}(\Gamma) - \{e\}$. In this case there is a morphism $\overline{M}(\Upsilon) : \overline{M}_{g,n,\Gamma} \rightarrow \overline{M}_{g,n,\Gamma'}$ obtained by forgetting the corresponding marking and collapsing any unstable components.

Remark 2.4. A morphism $\Upsilon : \Gamma \rightarrow \Gamma'$ which collapses an edge or loop induces an inclusion $\overline{M}(\Upsilon)$ of $\overline{M}_{g,n,\Gamma} \rightarrow \overline{M}_{g,n,\Gamma'}$. The morphisms arising in this way define on the collection of spaces $(\overline{M}_{g,n})_{2g+n \geq 3}$ the structure of a *modular operad* [30]; however, we will not need this language.

Proposition 2.5. *The boundary of $M_{g,n,\Gamma}$ in $\overline{M}_{g,n,\Gamma}$ is the union of spaces $M_{g,n,\Gamma'}$ such that Γ is obtained from Γ' by collapsing edges and forgetting loops. The boundary of $\overline{M}_{g,n}$ is a union of the following subspaces (which will be divisors with respect to the algebraic structure of the moduli space introduced later)*

- (a) (Non-separating node) if $2g + n > 3$, a subspace

$$\iota_{g-1,n+2} : D_{g-1,n+2} \rightarrow \overline{M}_{g,n}$$

equipped with an isomorphism

$$\varphi_{g-1,n+2} : D_{g-1,n+2} \rightarrow \overline{M}_{g-1,n+2}.$$

The inclusion is obtained by identifying the last two marked points.

- (b) (Separating node) for each splitting $g = g_1 + g_2, \{1, \dots, n\} = I_1 \cup I_2$ with $2g_j + |I_j| \geq 3, j = 1, 2$, a subspace

$$\iota_{g_1+g_2, I_1 \cup I_2} : D_{g_1+g_2, I_1 \cup I_2} \rightarrow \overline{M}_{g,n}$$

corresponding to the formation of a separating node, splitting the surface into pieces of genus g_1, g_2 with markings I_1, I_2 , equipped with an isomorphism

$$\varphi_{g_1+g_2, I_1 \cup I_2} : D_{g_1+g_2, I_1 \cup I_2} \rightarrow \overline{M}_{g_1, |I_1|+1} \times \overline{M}_{g_2, |I_2|+1}$$

(except that on the level of orbifolds in the case $I_1 = I_2 = \emptyset$ and $g_1 = g_2$ there is an additional automorphism exchanging the components.)

The pull-back $\iota_{g_1+g_2, I_1 \cup I_2}^* \beta$ of any class $\beta \in H(\overline{M}_{g,n})$ has a Künneth decomposition

$$(2) \quad \iota_{g_1+g_2, I_1 \cup I_2}^* \beta = \sum_{j \in J} \beta_{1,j} \otimes \beta_{2,j}$$

for some index set J and classes $\beta_{k,j} \in H(\overline{M}_{g_k, |I_k|+1})$. The moduli spaces $\overline{M}_{g,n}$ have the structure of orbifolds, with the automorphism group of an element given by the automorphism group of the modular graph giving its combinatorial type. In particular, each boundary divisor $D_{g-1, n+2}$ or $D_{g_1+g_2, I_1 \cup I_2}$ has a homology class $\overline{M}_{g,n}$, and since $\overline{M}_{g,n}$ is a rational homology manifold each of these homology classes has a dual class $\gamma_{g-1, n+2}$ resp. $\gamma_{g_1+g_2, I_1 \cup I_2}$ in $H^2(\overline{M}_{g,n}, \mathbb{Q})$. For the following, see Manin [57].

Definition 2.6. A *cohomological field theory* (CohFT) with values in a field Λ is a \mathbb{Z}_2 -graded vector space V equipped with a symmetric non-degenerate bilinear form $V \times V \rightarrow \Lambda$ and collection of S_n -invariant (with Koszul signs) *correlators*

$$V^n \times H(\overline{M}_{g,n}) \rightarrow \Lambda, \quad (\alpha, \beta) \mapsto \langle \alpha; \beta \rangle_{g,n}, \quad g, n \geq 0$$

where by convention $H(\overline{M}_{g,n}) = \Lambda$ if $\overline{M}_{g,n} = \emptyset$, satisfying the following two splitting axioms:

- (a) (Non-separating node) if $g \geq 1$ then

$$\langle \alpha; \beta \cup \gamma_{g-1, n+2} \rangle_{g,n} = \sum_k \langle \alpha, \delta_k, \delta^k; \iota_{g-1, n+2}^* \beta \rangle_{g-1, n+2}$$

where δ_k, δ^k are dual bases for V ;

- (b) (Separating node) if $2g + n \geq 4$, $I_1 \cup I_2$ is a partition of $\{1, \dots, n\}$, and $g = g_1 + g_2$ with $2g_i + |I_i| \geq 3$ for $i = 1, 2$ then

$$\langle \alpha; \beta \cup \gamma_{g_1+g_2, I_1 \cup I_2} \rangle_{g,n} = \sum_k \langle (\alpha_i)_{i \in I_1}, \delta_k; \cdot \rangle_{g_1, |I_1|+1} \langle (\alpha_i)_{i \in I_2}, \delta^k; \cdot \rangle_{g_2, |I_2|+1} (\iota_{g_1+g_2, I_1 \cup I_2}^* \beta)$$

where the dots indicate insertion of the Künneth components of $\iota_{g_1+g_2, I_1 \cup I_2}^* \beta$, ($\delta_k \in V_{k=1}^{\dim(V)}$ is a basis and $(\delta^k \in V)_{k=1}^{\dim(V)}$ a dual basis, and there is an additional factor of 2 in the exceptional case $g_1 = g_2, I_1 = I_2 = \emptyset$ arising from the additional automorphism.

That is, if β is as in (2) then

$$\langle \alpha; \beta \cup \gamma_{g_1+g_2, I_1 \cup I_2} \rangle_{g,n} = \sum_{j \in J, k} \langle (\alpha_i)_{i \in I_1}, \delta_k; \beta_{1,j} \rangle_{g_1, |I_1|+1} \langle (\alpha_i)_{i \in I_2}, \delta^k; \beta_{2,j} \rangle_{g_2, |I_2|+1}.$$

The correlators of any cohomological field theory define a family of associative algebra structures. In the standard axiomatization these are part of the associated *Frobenius manifold structure* on V [57]. However for the purposes of functoriality it is helpful to keep the “algebra” and “metric” parts of this structure separate, and instead we define the following:

Definition 2.7. A *CohFT algebra* consists of a \mathbb{Z}_2 -graded vector space V and a collection of S_n -invariant (with Koszul signs) multilinear maps

$$\mu^n : V^n \times H(\overline{M}_{0,n+1}) \rightarrow V, n \geq 2$$

satisfying the splitting axiom for any subset $I \subset \{1, \dots, n\}$ of order at least two:

$$(3) \quad \mu^n(\alpha_1, \dots, \alpha_n; \gamma_{0, I \cup (I^c \cup \{0\})} \cup \beta) = \sum_j \mu^{n-|I|+1}(\alpha_i, i \notin I, \mu^{|I|}(\alpha_i, i \in I; \beta_{1,j}); \beta_{2,j})$$

where $\iota_I^* \beta = \sum_j \beta_{1,j} \otimes \beta_{2,j}$ is the Künneth decomposition of the restriction of β as in (2).

Proposition 2.8. Any CohFT gives rise to a CohFT algebra.

Proof. By restricting to genus zero, and using duality to put one factor of V on the right. The splitting axiom (3) is a special case of the splitting axiom in Definition 2.6. \square

Remark 2.9. (a) Manin [57] terms a set of such maps a Comm_∞ -structure on V . However we avoid this terminology since it isn't clear what this is a homotopy version of.

(b) Note that the CohFT algebra structure does not require a metric, so a CohFT may be thought of roughly speaking as a CohFT algebra plus a metric. We remark that a more natural definition of CohFT may be obtained by allowing incoming and outgoing markings. However, we prefer to write the classes on the left to emphasize the analogy with A_∞ spaces.

(c) The various relations in $\overline{M}_{0,n+1}$ give rise to relations on the maps μ^n . In particular the map $\mu^2 : V \times V \rightarrow V$ is associative, by the splitting axiom and the relation

$$[D_{0,\{0,3\} \cup \{1,2\}}] = [D_{0,\{0,1\} \cup \{2,3\}}] \in H^2(\overline{M}_{0,4}).$$

(d) The notion of CohFT algebra is the “complex analog” of the notion of A_∞ algebra in the following imprecise sense. Denote by $\overline{M}_{0,n}^{\mathbb{R}}$ the moduli space of projective nodal curves C equipped with an anti-holomorphic involution fixing the markings, that is, the moduli space of *stable n -marked disks* where the markings are not necessarily in cyclic order. The symmetric group S_n acts canonically on $\overline{M}_{0,n}^{\mathbb{R}}$ by permuting the markings. The quotient of $\overline{M}_{0,n}^{\mathbb{R}}$ by the action of S_n is homeomorphic to the $n-1$ -st associahedron Assoc_{n-1} introduced in Stasheff [86]. An A_∞ space is a space X equipped with a collection of maps

$$X^n \times \text{Assoc}_n \rightarrow X, n \geq 2$$

satisfying a splitting axiom for the restriction to the boundary, and A_∞ algebras arise as spaces of chains on A_∞ spaces. To obtain the notion of CohFT algebra we replace X by a vector space and Assoc_n by the cohomology of its complexification. One could also imagine a cochain-level version but there are reasons to expect that this gives nothing new, see Teleman [88]

- (e) The notion of a genus zero CohFT can be repackaged in terms of a non-linear structure called a *Frobenius manifold*, which is a non-linear generalization of the notion of *Frobenius algebra* (unital algebra with compatible metric.) Any genus zero CohFT (with unit and grading) gives rise to a Frobenius manifold whose potential is

$$f : V \rightarrow \Lambda, \quad f(v) = \sum_{n \geq 3} \frac{1}{n!} \langle v, \dots, v; 1 \rangle_{0,n}.$$

The third derivatives of f give rise to a family of algebra structures

$$\star_v : T_v V^2 \rightarrow T_v V.$$

These give rise to a family of connections depending on a parameter \hbar ,

$$\nabla_{\hbar} : \Omega^0(V, TV) \rightarrow \Omega^1(V, TV), \quad \nabla_{\hbar, \xi} \sigma(v) = (d\sigma(\xi))(v) - (1/\hbar) \xi \star_v \sigma(v).$$

The associativity of \star translates into the flatness of ∇_{\hbar} , so that locally there exist sections $\sigma : V \rightarrow TV$ satisfying

$$(4) \quad (\text{Quantum Differential Equation}) \quad \hbar \partial_{\xi} \sigma(v) = \xi \star_v \sigma, \forall v, \xi \in V.$$

For a full discussion of the correspondence between Frobenius manifolds and CohFT's the reader is referred to Manin [57].

2.2. Complexified cyclohedron and traces on CohFT algebras. In this section we study the moduli spaces of stable marked *parametrized* curves. These are a special case of moduli spaces of stable maps (the degree one case) but we prefer to view them in a different way, as a special case of the Fulton-MacPherson construction [28]. We then discuss the associated notion of *trace* on a CohFT algebra. Let C be a smooth connected projective curve.

- Definition 2.10.** (a) (Parametrized nodal curves) A C -*parametrized nodal curve* is a (possibly disconnected) nodal curve \hat{C} equipped with a morphism $u : \hat{C} \rightarrow C$ of homology class $u_*[\hat{C}] = [C]$. That is, \hat{C} is the union of irreducible components C_0, \dots, C_r where u maps the *principal component* C_0 isomorphically onto C and u maps the other irreducible components C_1, \dots, C_r onto points. A *marking* of a C -parametrized curve is an n -tuple $\underline{z} = (z_1, \dots, z_n)$ of points in C^n distinct from the nodes and each other. An *isomorphism* of such curves is an isomorphism of nodal curves which induces the identity on C .
- (b) (Stable parametrized curves) A C -parametrized curve is *stable* if it has no infinitesimal automorphisms, that is, each non-principal irreducible component of C has at least three marked or nodal points.
- (c) (Rooted forests) Any C -parametrized curve has a *combinatorial type* which is a forest Γ (finite collection of trees) with a distinguished *root vertex* corresponding to the principal component and a labelling of the semiinfinite edges given by a bijection $l : \text{Edge}_{\infty}(\Gamma) \rightarrow \{1, \dots, n\}$. A rooted forest is *stable* if it corresponds to a stable parametrized curve, that is, each non-root vertex has valence at least three.

The set $\overline{M}_n(C)$ of isomorphism classes of connected stable C -parametrized curves has a natural topology, similar to that of $\overline{M}_{0,n}$ in genus zero: The following can be taken as a definition or a proposition using a suitable construction of the universal deformation of a stable map to C :

Definition 2.11. (Convergence of a sequence of parametrized stable curves) Suppose C has genus 0. A sequence $[(\hat{C}_\nu, u_\nu)]$ with smooth domain \hat{C}_ν converges to $[(\hat{C}, u)]$ if there exists, for each irreducible component \hat{C}_j of the limit \hat{C} , a sequence of holomorphic isomorphisms $\phi_{j,\nu} : \hat{C}_j \rightarrow \hat{C}_\nu$ such that

- (a) (Limit of a marking) for all i, j , $\lim_{\nu \rightarrow \infty} \phi_{j,\nu}^{-1}(z_{i,\nu}) = z_i^j$, the node in C_j connecting to the irreducible component of C containing z_i , or z_i if z_i is contained in C_j ;
- (b) (Limit of a different parametrization) for all $j \neq k$, $\lim_{\nu \rightarrow \infty} \phi_{j,\nu} \phi_{k,\nu}^{-1}$ has limit the constant map with value the node of C_j connecting to C_k ; and
- (c) (Limit of the map) for all j , $\lim \phi_{j,\nu}^* u_\nu = u|_{C_j}$.

Convergence for nodal domains \hat{C}_ν is defined similarly, by considering convergence on each irreducible component separately.

The definition for arbitrary genus is similar, but the maps $\phi_{j,\nu}$ exist only after removing small neighborhoods of the nodes. The topology on $\overline{M}_n(C)$ induced by this notion of convergence is compact and Hausdorff, and a special case of the Fulton-MacPherson compactification of configuration spaces considered in [28]. The open stratum $M_n(C)$ of \overline{M}_n is the configuration space $\text{Conf}_n(C)$ of n -tuples of distinct points on C .

More generally for any rooted forest Γ with n semiinfinite edges we denote by $M_{n,\Gamma}(C)$ the moduli space of isomorphism classes with type Γ and by $\overline{M}_{n,\Gamma}(C)$ its closure. The moduli spaces of stable marked curves $\overline{M}_{n,\Gamma}(C)$ satisfy a natural functoriality with respect to morphisms of rooted forests Γ . We say that a *morphism of rooted forests* is a morphism of modular graphs corresponding to the rooted forests mapping the root vertex to the root vertex.

Proposition 2.12. (Morphisms of moduli spaces associated to morphisms of forests)

- (a) Any morphism of rooted forests $\Upsilon : \Gamma \rightarrow \Gamma'$ induces a morphism of moduli spaces $\overline{M}(\Upsilon) : \overline{M}_{n,\Gamma}(C) \rightarrow \overline{M}_{n,\Gamma'}(C)$.
- (b) The boundary of $M_{n,\Gamma}(C)$ is the union of spaces $M_{n,\Gamma'}(C)$ such that there is a morphism of rooted forests $\Gamma' \rightarrow \Gamma$ collapsing an edge.
- (c) If Γ' is obtained from Γ by cutting an edge, then there is an isomorphism $\overline{M}_{n,\Gamma'}(C) \rightarrow \overline{M}_{n,\Gamma}(C)$ identifying the vertices corresponding to the additional semi-infinite edges.

The proof is standard from properties of moduli spaces of stable maps. Note that if $\Gamma' = \Gamma_0 \cup \Gamma_1$ is disconnected where Γ_0 contains the root vertex then $\overline{M}_{n,\Gamma}(C) \cong \overline{M}_{n_0,\Gamma_0}(C) \times \overline{M}_{0,n_1,\Gamma_1}$ where n_j is the number of semi-infinite edges of Γ_j . Thus the boundary of $\overline{M}_{n,\Gamma}$ is the union of products of lower-dimensional moduli spaces of C -parametrized stable curves and stable curves.

Remark 2.13. (Relation to the cyclohedron) Let C be a projective line. Any anti-holomorphic involution of C induces an anti-holomorphic involution of $\overline{M}_n(C)$, with fixed point set $\overline{M}_n(C)^{\mathbb{R}}$ identified with the moduli space of stable parametrized n -marked disks. The symmetric group S_n acts by permutation, and the quotient by S_{n-1} is isomorphic to the subset $\overline{M}_n(C)^{\mathbb{R},+}$ of $\overline{M}_n(C)^{\mathbb{R}}$ such that the marked points z_0, \dots, z_n occur in cyclic order around the boundary of the disk. The action of S^1 by rotation preserves $\overline{M}_n(C)^{\mathbb{R},+}$ and the quotient is the cyclohedron Cycl_n , that is, the moduli space of points on the circle compactified by allowing bubbling, see Markl [59]. In this sense it is slight abuse of terminology to call $\overline{M}_n(C)$ the complexification

of Cycl_n ; rather, $\overline{M}_n(C)$ is the complexification of a circle bundle over Cycl_n . This dishonesty will be somewhat remedied in the last section when we consider $\overline{M}_n(C)$ with its circle action.

The boundary structure of the moduli space $\overline{M}_n(C)$ is described in the following.

Proposition 2.14. *The boundary of $\overline{M}_n(C)$ is the union of the following subspaces (which will be divisors once the algebraic structure on $\overline{M}_n(C)$ is introduced): For each subset $I \subset \{1, \dots, n\}$ of order at least two a subspace $\iota_I : D_I \rightarrow \overline{M}_n(C)$ where the markings for $i \in I$ have bubbled off onto an (unparametrized) sphere bubble. The subspace D_I admits a homeomorphism $\varphi_I : D_I \rightarrow \overline{M}_{0,|I|+1} \times \overline{M}_{n-|I|+1}(C)$.*

For any $\beta \in \overline{M}_n(C)$, the pull-back $\iota_I^* \beta$ to a subspace D_I has a Künneth decomposition

$$(5) \quad \iota_I^* \beta = \sum_{j \in J} \beta_{1,j} \otimes \beta_{2,j}$$

for some index set J and classes $\beta_{1,j} \in H(\overline{M}_{0,|I|+1})$ and $\beta_{2,j} \in \overline{M}_{n-|I|+1}(C)$. In general, the moduli space of stable maps is not smooth. However, the space $\overline{M}_n(C)$, as a special case of Fulton-MacPherson [28], is a compact smooth manifold. In particular, any subset D_I has a homology class $[D_I] \in H_2(\overline{M}_n(C), \mathbb{Z})$ and a dual class $\gamma_I \in H^2(\overline{M}_n(C), \mathbb{Z})$, although we work with rational coefficients below. Let Λ be a vector space.

Definition 2.15. (Trace on a CohFT algebra) A (C -based, Λ -valued) *trace* on a CohFT algebra V is a collection of S_n -invariant (with Koszul signs) multilinear maps

$$\tau^n : V^n \times H(\overline{M}_n(C)) \rightarrow \Lambda, \quad n \geq 0$$

satisfying a splitting axiom

$$\tau^n(\alpha; \beta \cup \gamma_I) = \tau^{n-|I|+1}(\alpha_i, i \notin I, \mu^{|I|}(\alpha_i, i \in I; \cdot); \cdot)(\iota_I^* \beta)$$

where γ_I is the dual class to D_I and the \cdot 's denote insertion of the Künneth components of β . That is, with β as in (5),

$$\tau^n(\alpha; \beta \cup \gamma_I) = \sum_{j \in J} \tau^{n-|I|+1}(\alpha_i, i \notin I, \mu^{|I|}(\alpha_i, i \in I; \beta_{1,j}); \beta_{2,j}).$$

Remark 2.16. (a) In our main application, gauged Gromov-Witten invariants will define a Λ_X^G -valued trace on $QH_G(X)$, which is a CohFT defined over the field of fractions of $H(BG) \otimes \Lambda_X^G$. In other words, the space Λ above need not be the ring or field over which the CohFT algebra or CohFT is defined.

(b) One should compare the notion of trace with that for A_∞ algebras described in [58, Proposition 2.14]. The corresponding notion for an A_∞ space X consists of a sequence of maps

$$X^n \times \text{Cycl}_n \rightarrow Y, \quad n \geq 0$$

to an ordinary space Y , satisfying a suitable splitting axiom.

Any trace $(\tau^n)_{n \geq 0}$ on a CohFT algebra V defines a formal map

$$(6) \quad \tau : V \rightarrow \Lambda, \quad v \mapsto \sum_n \frac{1}{n!} \tau^n(v, \dots, v)$$

often called a *potential*. The splitting axiom implies that the second derivatives of τ with two point classes inserted define a Λ -valued family of bilinear forms on $T_v V$ compatible with the multiplications \star_v on $T_v V$:

Definition 2.17. (Family of bilinear forms associated to a CohFT trace) Let $\gamma_j \in \overline{M}_n(C)$ denote the class given by the pullback of a point under the j -th evaluation map $\text{ev}_j : \overline{M}_n(C) \rightarrow C$. For $v, v_1, v_2 \in V$ define

$$g_v : T_v V^2 \rightarrow \Lambda, \quad g_v(v_1, v_2) := \sum_{n \geq 0} \frac{1}{n!} \tau^{n+2}(v_1, v_2, v, \dots, v; \gamma_1 \cup \gamma_2).$$

the *bilinear form* on $T_v V$ associated to the trace.

Remark 2.18. (Why fix two points?) The additional insertion of $\gamma_1 \cup \gamma_2$ is designed so that the leading order term is associated to the moduli space of parametrized 2-marked curves with the points fixed, so that in e.g. Gromov-Witten theory one gets the usual metric. An alternative would be to define a family of *algebra traces* $V \rightarrow \Lambda$ associated to τ , but it seems here that one still has to fix the point to get the classical notion in Gromov-Witten theory.

Proposition 2.19. (Compatibility of the family of bilinear forms with the family of products) Let $(\tau^n)_{n \geq 0}$ be a trace on a CohFT algebra V . For $v, v_1, v_2, v_3 \in V$, we have $g_v(v_1 \star_v v_2, v_3) = g_v(v_1, v_2 \star_v v_3)$.

Proof. The relation in homology of the 6-dimensional space $\overline{M}_3(C)$

$$[D_{\{1,3\}}] \cup \gamma_1 \cup \gamma_2 = [D_{\{2,3\}}] \cup \gamma_1 \cup \gamma_2 \in H^6(\overline{M}_3(C))$$

pulls back to a relation in $H(\overline{M}_n(C))$, ≥ 4 ,

$$\sum_{I \subset \{4, \dots, n\}} [D_{\{1,3\} \cup I}] \cup \gamma_1 \cup \gamma_2 = \sum_{I \subset \{4, \dots, n\}} [D_{\{2,3\} \cup I}] \cup \gamma_1 \cup \gamma_2 \in H^6(\overline{M}_n(C))$$

The claim now follows from the splitting axiom in 2.15. \square

2.3. Complexified multiplihedron and morphisms of CohFT algebras. Ma'u-Woodward [61], based on earlier work of Ziltener [98], introduces a compactification of the moduli space of distinct points on the affine line up to translation, which “complexifies” the multiplihedron in the same way that the Grothendieck-Knudsen space and the Fulton-MacPherson spaces complexify the associahedron and cyclohedron respectively. We then discuss the associated notion of *morphism* of CohFT algebras. Let \mathbb{A} denote an affine line over \mathbb{C} , unique up to isomorphism. We denote by

$$\Omega^1(\mathbb{A}, \mathbb{C})^{\mathbb{C}} = \{wdz | w \in \mathbb{C}\} \cong \mathbb{C}$$

the space of \mathbb{C} -invariant one-forms on \mathbb{A} .

Definition 2.20. (Scaled affine line) A *scaling* of an affine line \mathbb{A} is a translation-invariant, non-zero one form $\lambda \in \Omega^1(\mathbb{A}, \mathbb{C})^{\mathbb{C}}$. A *scaled affine line* is an affine line equipped with a scaling. An *n-marking* of an affine line is an n -tuple $\underline{z} = (z_1, \dots, z_n)$ of distinct points in \mathbb{A}^n . An *isomorphism* of scaled n -marked affine lines is an affine isomorphism $\psi : C_0 \rightarrow C_1$, such that $\psi^* \lambda_1 = \lambda_0$ and $\psi(z_{0,i}) = z_{1,i}$, $i = 1, \dots, n$.

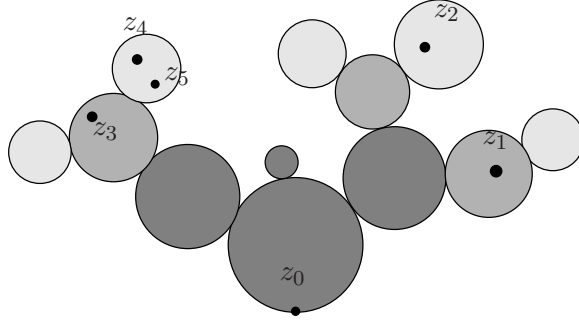


FIGURE 4. An affine scaled curve

Let $M_{n,1}(\mathbb{A})$ denote the moduli space of isomorphism classes of scaled n -marked affine lines. If \mathbb{A} is a scaled affine line then the group of automorphisms of \mathbb{A} preserving the scaling is the additive group \mathbb{C} acting on \mathbb{A} by translation. Thus

Proposition 2.21. *The moduli space $M_{n,1}(\mathbb{A})$ may be identified with the configuration space $\text{Conf}_n(\mathbb{A})$ of n -tuples of distinct points on \mathbb{A} up to the action of \mathbb{C} by translation, $M_{n,1}(\mathbb{A}) \cong \text{Conf}_n(\mathbb{A})/\mathbb{C}$.*

Proof. For any tuple $(z_1, \dots, z_n, \lambda)$ we take the unique rescaling so that $\lambda = dz$ and then take the associated configuration. Conversely, any configuration defines an affine scaled map by taking the scaling $\lambda = dz$ to be standard. \square

Remark 2.22. (Two-forms instead of one-forms) The moduli space $M_{n,1}(\mathbb{A})$ can be viewed in a different way: Any scaling λ gives rise to a real area form $\omega_{\mathbb{A}} := \lambda \wedge \bar{\lambda}$ on \mathbb{A} . Replacing λ with $\omega_{\mathbb{A}}$ amounts to forgetting a complex phase; thus, one can view $M_{n,1}(\mathbb{A})$ as the moduli space of data $(z_1, \dots, z_n, \omega, \phi)$ where $z_1, \dots, z_n \in \mathbb{A}$ are distinct points, $\omega_{\mathbb{A}} \in \Omega^2(\mathbb{A}, \mathbb{R})$ is a translationally-invariant area form, and $\phi \in U(1)$ is a phase. Any automorphism ψ of \mathbb{A} has a well-defined *argument* $\arg(\psi) \in U(1)$ giving the angle of rotation, and ψ acts on $(\omega_{\mathbb{A}}, \phi)$ by $(\psi^*\omega_{\mathbb{A}}, \arg(\psi)\phi)$. This is the point of view taken in Ziltener's thesis [98] who suggested the motto “one cannot rotate affine symplectic vortices”.

The moduli space $M_{n,1}(\mathbb{A})$ has a natural compactification obtained by allowing bubbles with degenerate scalings.

Definition 2.23. (Stable nodal scaled affine lines) A *possibly degenerate scaling* on an affine line \mathbb{A} is an element of the set

$$\overline{\Omega}^1(\mathbb{A}, \mathbb{C})^{\mathbb{C}} = \Omega^1(\mathbb{A}, \mathbb{C})^{\mathbb{C}} \cup \{\infty\} \cong \mathbb{P}.$$

A possibly degenerate scaling λ is *degenerate* if $\lambda = 0$ or $\lambda = \infty$ and is *non-degenerate* otherwise. The action of the group of automorphisms $\text{Aut}(\mathbb{A})$ on $\Omega^1(\mathbb{A}, \mathbb{C})^{\mathbb{C}}$ by pull-back extends naturally to an action on $\overline{\Omega}^1(\mathbb{A}, \mathbb{C})$, with fixed points $\{0\}, \{\infty\}$. A *nodal marked scaled affine line* is a datum $(C, z_0, \dots, z_n, \lambda)$ satisfying the following condition:

(Monotonicity) on any non-self-crossing path from a marking z_i to the root marking z_0 , there is exactly one *colored irreducible component* with finite scaling; and the irreducible

components before (resp. after) this irreducible component have infinite (resp. zero scaling).

See Figure 4, where irreducible components with infinite resp. finite, non-zero resp. zero scaling are shown with dark resp. grey resp. light grey shading. An *isomorphism* of nodal marked scaled affine lines (C_j, z_j, λ_j) , $j = 0, 1$ is an isomorphism of nodal curves $\phi : C_0 \rightarrow C_1$ intertwining the (possibly degenerate) scalings and markings in the sense that $\phi^* \lambda_1 = \lambda_0$ and $\phi(z_{0,i}) = z_{1,i}$. A nodal marked scaled affine line is *stable* if it has no automorphisms, or equivalently, if each irreducible component with finite scaling has at least two special points, and each irreducible component with degenerate scaling has at least three special points.

The space $\overline{M}_{n,1}(\mathbb{A})$ of isomorphism classes of connected stable scaled n -marked lines has a natural topology, similar to the topology on the moduli space of stable curves. Given a stable affine scaled curve $(C, z_1, \dots, z_n, \lambda)$, for any marking z_i and irreducible component C_j we denote by z_i^j the node in C_j connecting to the irreducible component of C containing z_i , or z_i if z_i is contained in C_j . The following can be taken as a definition or a proposition with a suitable notion of family of stable scaled marked lines, see Example 4.2.

Definition 2.24. (Convergence of a sequence of nodal scaled affine lines) A sequence $[(C_\nu, z_{1,\nu}, \dots, z_{n,\nu}, \lambda_\nu)]$ with smooth domain C_ν *converges* to $[(C, z_1, \dots, z_n, \lambda)]$ if there exists, for each irreducible component C_j of the limit C , a sequence of holomorphic isomorphisms $\phi_{j,\nu} : C_j \rightarrow C_\nu$ such that

- (a) (Limit of the scaling) $\lim_{\nu \rightarrow \infty} \phi_{j,\nu}^* \lambda_\nu = \lambda|_{C_j}$
- (b) (Limit of a marking) $\lim_{\nu \rightarrow \infty} \phi_{j,\nu}^{-1}(z_{i,\nu}) = z_i^j$
- (c) (Limit of a different parametrization) $\lim_{\nu \rightarrow \infty} \phi_{j,\nu} \phi_{i,\nu}^{-1}$ has limit the constant map with value the node of C_j connecting to C_i .

Convergence for sequences with nodal domain is defined similarly.

Example 2.25. (Two markings converging) If $C_\nu = \mathbb{P} = \mathbb{A} \cup \{\infty\}$ and two points $z_{i,\nu}, z_{j,\nu}$ come together in the sense that $\lim_{\nu \rightarrow \infty} z_{i,\nu} - z_{j,\nu} \rightarrow 0$, then there exists a sequence of holomorphic maps $\phi_\nu : \mathbb{P} \rightarrow C_\nu$ such that $\phi_\nu^{-1}(z_{i,\nu}), \phi_\nu^{-1}(z_{j,\nu})$ converge to distinct points, and the scaling $\phi_\nu^* \lambda_\nu$ converges to zero. The limiting configuration consists of a irreducible component with two markings and one node with zero scaling, and an irreducible component with finite scaling, one node, and the root marking z_0 . See Figure 5.

Example 2.26. (Two markings diverging) If $C_\nu = \mathbb{P}$ for all ν with constant scaling λ_ν and two points $z_{i,\nu}, z_{j,\nu}$ go to infinity in $\mathbb{A} \subset \mathbb{P}$ in different directions, then for $k \in \{i, j\}$ there exists (i) a sequence of holomorphic maps $\phi_{k,\nu} : \mathbb{C} \rightarrow C_\nu$ such that $\phi_{k,\nu}^{-1}(z_{k,\nu})$ and $\phi_{k,\nu}^* \lambda_\nu$ converge and (ii) a sequence $\phi_{ij,\nu} : \mathbb{C} \rightarrow C_\nu$ such that $\phi_{ij,\nu}^{-1} z_{k,\nu}$ for $k = i, j$ converge to distinct points and $\phi_{ij,\nu}^* \lambda_\nu$ converges to infinity. The limiting configuration consists of two components with a single marking and node and finite scaling, and a component with two nodes, the root marking, and infinite scaling. See Figure 6.

We denote by $\overline{M}_{n,1}(\mathbb{A})$ the space of isomorphism classes of connected nodal scaled lines, equipped with the topology above. By Ma'u-Woodward [61] $\overline{M}_{n,1}(\mathbb{A})$ is a compact Hausdorff space.

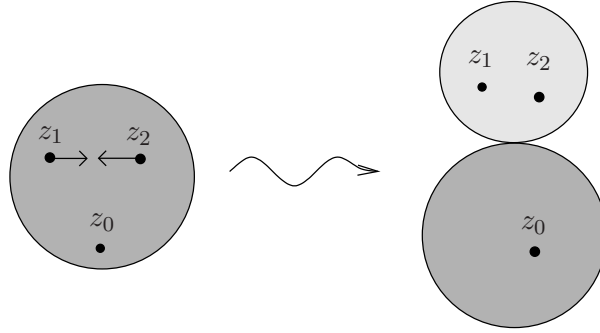


FIGURE 5. Two markings converging

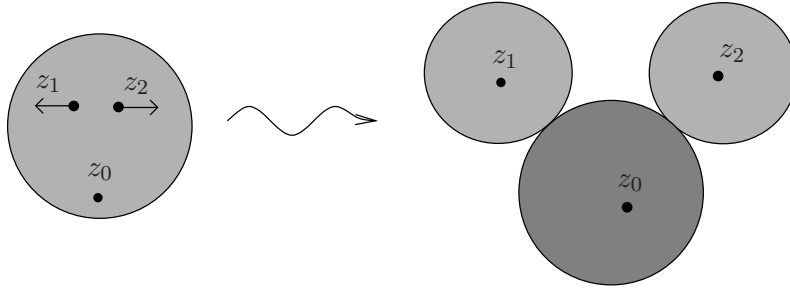


FIGURE 6. Two markings diverging

Definition 2.27. (Combinatorial types of nodal scaled affine lines) The *combinatorial type* of a connected scaled affine line is a *colored tree* consisting of a tree $\Gamma = (\text{Vert}(\Gamma), \text{Edge}(\Gamma))$ together with a partition of the vertices

$$\text{Vert}(\Gamma) = \text{Vert}^0(\Gamma) \cup \text{Vert}^1(\Gamma) \cup \text{Vert}^\infty(\Gamma)$$

and a labelling of its semi-infinite edges given by a bijection $\text{Edge}_\infty(\Gamma) \rightarrow \{1, \dots, n\}$, and satisfying the combinatorial version of the monotonicity condition:

(Monotonicity) on any self-crossing path from a semi-infinite edge labelled j to the semi-infinite edge labelled 0, there is exactly one vertex in $\text{Vert}^1(\Gamma)$, all vertices before resp. after are in $\text{Vert}^0(\Gamma)$ resp. $\text{Vert}^\infty(\Gamma)$.

A colored tree is *stable* if it corresponds to a stable affine scaled line, that is, each vertex $v \in \text{Vert}^0(\Gamma)$ resp. $\text{Vert}^\infty(\Gamma)$ resp. $\text{Vert}^1(\Gamma)$ has valence at least 3 resp. 3 resp. 2.

We call the vertices in $\text{Vert}^1(\Gamma)$ the *colored vertices*. Colored trees can be pictured as trees where part of the tree containing the root semi-infinite edge is “below water” and part “above water”; the colored vertices in $\text{Vert}^1(\Gamma)$ are those “at the water level”. The monotonicity condition then says that the path from any “above water” semiinfinite edge to the unique “below water” semiinfinite edge passes through the water surface exactly once. However, in our trees we adopt the standard convention of having the root edge (which corresponds to an outgoing marking) at the top of the picture.

More generally we allow disconnected curves where each connected component is either a nodal affine curve, or a nodal curve with infinite or zero scaling.

Definition 2.28. (Colored forests) A colored forests Γ is a union of components that are either colored trees, or ordinary trees with all vertices in $\text{Vert}^0(\Gamma)$ or all vertices in $\text{Vert}^\infty(\Gamma)$.

Remark 2.29. The semi-infinite edges of a colored forest admit a partition $\text{Edge}(\Gamma) = \text{Edge}^0(\Gamma) \cup E^\infty(\Gamma)$ where $\text{Edge}^0(\Gamma)$ is the union of semi-infinite edges labelled $1, \dots, n$, and finite edges connecting $\text{Vert}^0(\Gamma)$ with $\text{Vert}^0(\Gamma) \cup \text{Vert}^1(\Gamma)$, and $E^\infty(\Gamma)$ is the union of the semi-infinite edge with the set of edges connecting $\text{Vert}^1(\Gamma) \cup \text{Vert}^\infty(\Gamma)$ with $\text{Vert}^\infty(\Gamma)$.

For any colored forest Γ with n semiinfinite edges we denote by $M_{n,1,\Gamma}(\mathbb{A})$ the space of isomorphism classes of scaled lines of combinatorial type Γ , and $\overline{M}_{n,1,\Gamma}(\mathbb{A})$ its closure.

Definition 2.30. (Morphisms of colored forests) A *morphism of colored forests* from Γ to Γ' is a combination of the following simple morphisms:

- (a) (Collapsing edges without relations) $\Upsilon : \Gamma \rightarrow \Gamma'$ *collapses an edge* if Υ is injective except that it maps two vertices in $\text{Vert}^0(\Gamma) \cup \text{Vert}^1(\Gamma)$ to the same vertex in $\text{Vert}(\Gamma')$ or two vertices in $\text{Vert}^\infty(\Gamma)$ with the same vertex in $\text{Vert}^\infty(\Gamma)$. (In other words, any edge except those connecting $\text{Vert}^1(\Gamma)$ with $\text{Vert}^\infty(\Gamma)$).

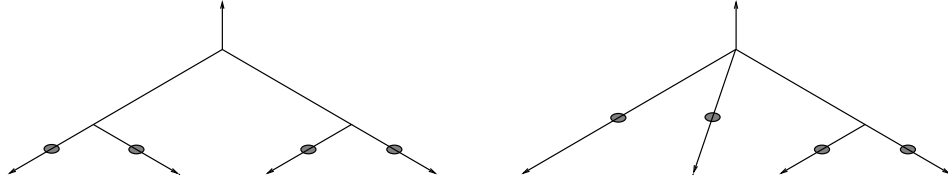


FIGURE 7. Collapsing an edge connecting two vertices of the same type

- (b) (Collapsing edges with relations) $\Upsilon : \Gamma \rightarrow \Gamma'$ *collapses edges* if Υ is injective except for one colored vertex of Γ' whose inverse image in $\text{Vert}(\Gamma)$ is a collection of colored vertices in Γ and a single vertex in $\text{Vert}^\infty(\Gamma)$, joined to each of the colored vertices by a single edge. Note that one cannot write such a morphism as a composition of morphisms each collapsing a single edge, since there is no way to assign the coloring of vertices of the resulting graph which results in a colored forest.

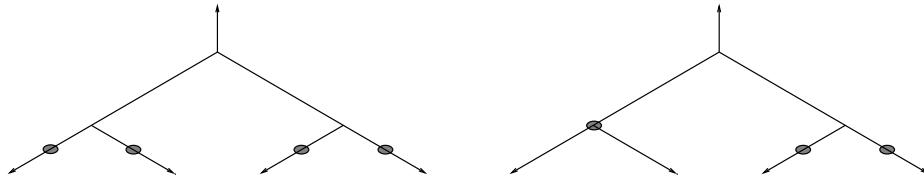


FIGURE 8. Collapsing edges with relations

- (c) (Cutting an edge without relations) $\Upsilon : \Gamma \rightarrow \Gamma'$ *cuts an edge* of Γ if the vertices are the same, but Γ' has one fewer edge than Γ and the edge does not lie between the colored vertices and the root edge.

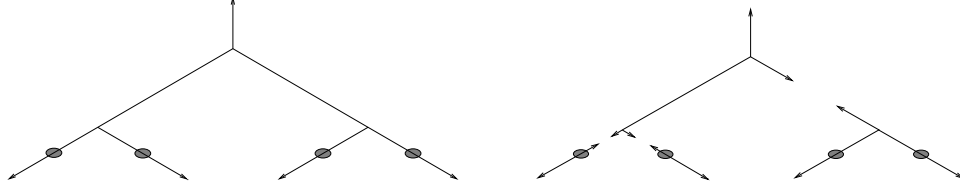


FIGURE 9. Cutting edges with relations

- (d) (Cutting edges with relations) $\Upsilon : \Gamma \rightarrow \Gamma'$ *cuts edges with relations* of Γ if the vertices are the same, but Γ' has fewer edges than Γ , with each removed edge lying on a path between the root edge and the colored vertices, and each path passing through a unique such edge. See Figure 9.
- (e) (Forgetting tails) $\Upsilon : \Gamma \rightarrow \Gamma'$ *forgets a tail* (semiinfinite edge) and any vertices that become unstable, recursively starting from the semiinfinite edges furthest away from the root edge.

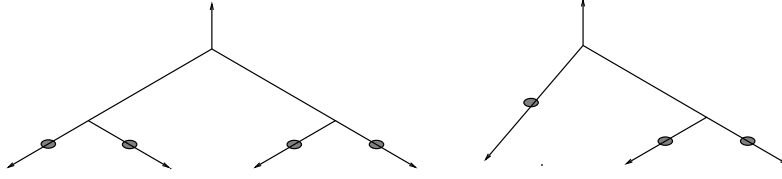


FIGURE 10. Forgetting a tail and collapsing

Remark 2.31. (More explanation on forgetting tails) Forgetting a tail leaves possibly only the vertex adjacent to the tail unstable, if it is colored with a single other edge adjacent, or non-colored with two other adjacent edges. In the first case, removing that vertex and the other adjacent edge still leaves a non-colored vertex which may be unstable, since it has one fewer edge. If unstable, removing this vertex and identifying the other two edges gives a stable colored tree. In the second case, removing the vertex gives a stable colored tree. See Figure 11.

Proposition 2.32. (Morphisms of moduli spaces induced by morphisms of colored forests) *To any morphism Υ of colored forests $\Gamma \rightarrow \Gamma'$ one can associate a morphism $\overline{M}_n(\Upsilon) : \overline{M}_{n,1,\Gamma}(\mathbb{A}) \rightarrow \overline{M}_{n,1,\Gamma'}(\mathbb{A})$ as follows.*

- (a) (Collapsing edges without relations) *Any morphism $\Upsilon : \Gamma \rightarrow \Gamma'$ collapsing an edge induces an inclusion $\overline{M}(\Upsilon) : \overline{M}_{n,1,\Gamma}(\mathbb{A}) \rightarrow \overline{M}_{n,1,\Gamma'}(\mathbb{A})$.*
- (b) (Collapsing edges with relations) *Any morphism $\Upsilon : \Gamma \rightarrow \Gamma'$ collapsing edges with relations induces an inclusion $\overline{M}(\Upsilon) : \overline{M}_{n,1,\Gamma}(\mathbb{A}) \rightarrow \overline{M}_{n,1,\Gamma'}(\mathbb{A})$.*
- (c) (Cutting an edge or edges with relations) *Any morphism $\Upsilon : \Gamma \rightarrow \Gamma'$ cutting an edge or edges with relations of Γ induces a homeomorphism from $\overline{M}_{n,1,\Gamma}(\mathbb{A})$ to $\overline{M}_{n,1,\Gamma'}(\mathbb{A})$ by identifying the markings corresponding to the additional semiinfinite edges.*
- (d) (Forgetting tails) *Any morphism $\Upsilon : \Gamma \rightarrow \Gamma'$ forgetting a tail induces a map $\overline{M}(\Upsilon) : \overline{M}_{n,1,\Gamma}(\mathbb{A}) \rightarrow \overline{M}_{n,1,\Gamma'}(\mathbb{A})$ which forgets the corresponding marking and collapses any*

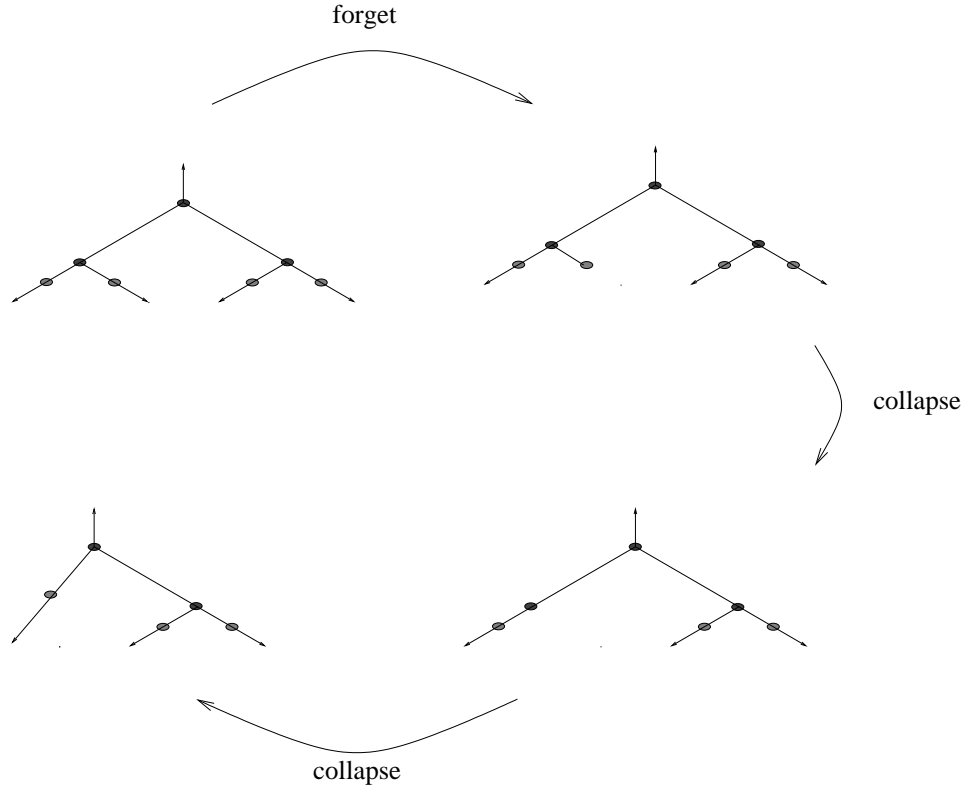


FIGURE 11. Collapsing unstable vertices (colored vertices are lightly shaded)

unstable components recursively starting with the semiinfinite edges corresponding to the finite markings.

Proof. The existence of these maps is immediate from the definitions except for the existence of the morphism for forgetting tails, which requires an inductive argument collapsing the unstable components. For this note that forgetting a tail leaves possibly only the component containing the corresponding marking unstable, if it had either non-degenerate scaling and two special points, or degenerate scaling and three special points. Removing that component, and if the second possibility holds, replacing the node with the remaining marking or identifying the two remaining nodes, produces a new curve with one fewer irreducible component. In the second case, the resulting curve is automatically stable. In the first case, the adjacent irreducible component has one fewer special point, and so now may be unstable. If so, removing that component, and either (i) identifying the nodes, if the two special points were nodes, or (ii) changing the node to a marking, if the two special points were a node and a marking, produces a stable scaled affine curve. We give a stronger, algebraic version of the forgetful morphism in Example 4.2. \square

Lemma 2.33. *For each Γ , the boundary of $\overline{M}_{n,1,\Gamma}(\mathbb{A})$ consists of those moduli spaces $M_{n,1,\Gamma'}(\mathbb{A})$ such that Γ is obtained from Γ' by collapsing an edge or edges with relations. Furthermore, each*

$\overline{M}_{n,1,\Gamma}(\mathbb{A})$ is a product of the moduli spaces $\overline{M}_{n_j,1}(\mathbb{A})$ and \overline{M}_{0,n_j} corresponding to the vertices of Γ , where n_j are the valences.

Woodward-Ma'u [61] shows that the compactification $\overline{M}_{n,1}(\mathbb{A})$ has the structure of a projective variety, locally isomorphic to a toric variety. The local structure of $\overline{M}_{n,1}(\mathbb{A})$ near the stratum $M_{n,1,\Gamma}(\mathbb{A})$ of nodal lines with combinatorial type Γ may be described as follows.

Definition 2.34. (Balanced labellings) For any colored tree Γ , a labelling $\gamma : \text{Edge}_{<\infty}(\Gamma) \rightarrow \mathbb{C}$ is *balanced* iff

$$(7) \quad \prod_{e \in P_{vw}} \gamma(e)^{\pm} = 1$$

where v, w range over elements of $\text{Vert}^1(\Gamma)$ and P_{vw} is the unique non self-crossing path from v to w , and in the product the sign is positive if e is pointing towards the root edge marked z_0 and negative otherwise.

Let $Z_\Gamma \subset \text{Map}(\text{Edge}_{<\infty}(\Gamma), \mathbb{C})$ denote the space of balanced labellings. An element of Z_Γ is called a tuple of *gluing parameters*.

Example 2.35. (A singularity in the moduli space) For the tree Γ in Figure 12, with large dots indicating vertices in $\text{Vert}^1(\Gamma)$, the relations are $\gamma_3 = \gamma_4, \gamma_1\gamma_3 = \gamma_2\gamma_5, \gamma_5 = \gamma_6$. The corresponding toric variety Z_Γ corresponds to a 3-dimensional cone with 4 extremal rays, and so the moduli space has a singularity at the vertex. Because toric surface singularities are always at worst orbifold singularities, the singularities of Z_Γ occur in complex codimension three and higher.

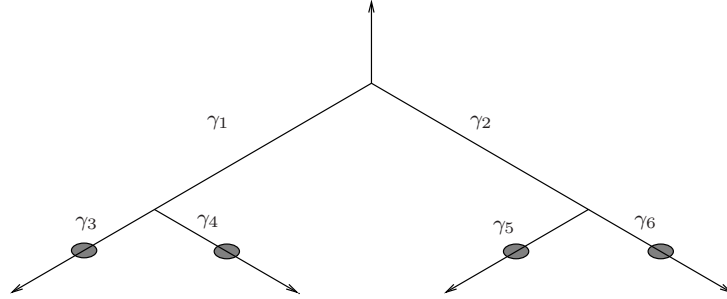


FIGURE 12. An example of a colored tree

Proposition 2.36. [61] *There exists an open neighborhood of $M_{n,1,\Gamma}(\mathbb{A})$ in $\overline{M}_{n,1}(\mathbb{A})$ isomorphic to an open neighborhood of $0 \times M_{n,1,\Gamma}(\mathbb{A})$ in $Z_\Gamma \times M_{n,1,\Gamma}(\mathbb{A})$.*

Proof. The construction is a version of the *Schiffer variation* construction of the universal deformation of a genus zero nodal curve [5, p. 176] in which small balls around the nodes are removed and the components glued together via maps $z \mapsto \gamma/z$. The scaling is determined by the product of the gluing parameters from the root component to the irreducible components with finite scaling, independent of the choice of irreducible component with finite scaling by the balanced condition (7). \square

Remark 2.37. (Codimension formula) The codimension of a stratum $\overline{M}_{n,1}(\mathbb{A})$ corresponding to a colored tree Γ is *not* the number of finite edges, but rather

$$\text{codim}(M_{n,1,\Gamma}(\mathbb{A})) = \# \text{Edge}_{<\infty}(\Gamma) + 1 - \# \text{Vert}^1(\Gamma)$$

where the extra summand $1 - \# \text{Vert}^1(\Gamma)$ corresponds to the minus the number of relations on the gluing parameters $(\gamma(e))_{e \in \text{Edge}_{<\infty}(\Gamma)} \in Z_\Gamma$.

Proposition 2.38. *The boundary of $\overline{M}_{n,1}(\mathbb{A})$ consists of the following subsets (which will be divisors with respect to the algebraic structure on $\overline{M}_{n,1}(\mathbb{A})$ introduced later):*

- (a) (Bubbling points) *For any $I \subset \{1, \dots, n\}$ of order at least two the subset*

$$\iota_I : D_I \rightarrow \overline{M}_{n,1}(\mathbb{A})$$

corresponding to the formation of a single bubble containing the markings I , with an isomorphism

$$(8) \quad D_I \rightarrow \overline{M}_{0,|I|+1} \times \overline{M}_{n-|I|+1,1}(\mathbb{A}).$$

- (b) (Blowing up scaling) *For any $r > 0$ and unordered partition $[I_1, \dots, I_r]$, $I_1 \cup \dots \cup I_r = \{1, \dots, n\}$ of order at least two, with each I_j non-empty, a subset $D_{[I_1, \dots, I_r]}$ corresponding to the formation of r bubbles with markings I_1, \dots, I_r , attached to a remaining component with infinite scaling. We have a homeomorphism*

$$(9) \quad D_{[I_1, \dots, I_r]} \cong \left(\prod_{i=1}^r \overline{M}_{|I_i|,1}(\mathbb{A}) \right) \times \overline{M}_{0,r}.$$

Remark 2.39. (a) The inclusions of these subspaces give the collection of spaces $\overline{M}_{n,1}(\mathbb{A})$ the structure of an algebra over the operad associated to the notion of *homotopy morphism* of operads in [60]. However, we will not use or need this language and will not discuss it further.

- (b) (Relation to the multiplihedron) The moduli space $\overline{M}_{n,1}(\mathbb{A})$ has a “positive real locus” that appears in Stasheff’s description of A_∞ morphisms [86]. Namely, taking a real structure on \mathbb{A} , the anti-holomorphic involution on \mathbb{A} induces an anti-holomorphic involution of $\overline{M}_{n,1}(\mathbb{A})$. We denote by $\overline{M}_{n,1}(\mathbb{A})^{\mathbb{R}}$ the fixed point locus, in which all markings are on the real line. The symmetric group S_n acts on $\overline{M}_{n,1}(\mathbb{A})$, and restricts to an action on $\overline{M}_{n,1}(\mathbb{A})^{\mathbb{R}}$ with fundamental domain given as the closure of the subset $M_{n,1}(\mathbb{A})^{\mathbb{R},+}$ where $z_1 < z_2 < \dots < z_n$, homeomorphic to Stasheff’s multiplihedron Mult_n [61]. An A_∞ morphism of A_∞ spaces X, Y consists of a sequence of maps

$$X^n \times \text{Mult}_n \rightarrow Y, n \geq 0$$

satisfying a suitable splitting axiom on the boundary.

The splitting axiom for morphisms of CohFT algebras is defined via divisors on $\overline{M}_{n,1}(\mathbb{A})$. Because the singularities of the toric variety Z_Γ occur in complex codimension at least three, $\overline{M}_{n,1}(\mathbb{A})$ has a unique homology class of top dimension. In particular, each of the boundary divisors above has a well-defined homology class in $\overline{M}_{n,1}(\mathbb{A})$. However, $\overline{M}_{n,1}(\mathbb{A})$ is not smooth

(and not a rational homology manifold) and not every boundary stratum has a dual class. That is, given a divisor

$$(10) \quad D = \sum_I n_I D_I + \sum_{r, [I_1, \dots, I_r]} n_{[I_1, \dots, I_r]} D_{[I_1, \dots, I_r]}$$

there may or may not exist a class $\gamma \in H^2(\overline{M}_{n,1}(\mathbb{A}))$ that satisfies

$$\langle \beta, [D] \rangle = \langle \beta \cup \gamma, [\overline{M}_{n,1}(\mathbb{A})] \rangle.$$

This requires the restriction to combinations of boundary divisors that have dual classes in the following definition.

Let $(V, (\mu_V^n)_{n \geq 2})$ and $(W, (\mu_W^n)_{n \geq 2})$ be CohFT algebras.

Definition 2.40. A *morphism of CohFT algebras* from V to W is a collection of S_n -invariant (with Koszul signs) multilinear maps

$$\phi^n : V^n \times H(\overline{M}_{n,1}(\mathbb{A})) \rightarrow W, \quad n \geq 0$$

such that for any divisor D of the form (10) with dual class $\gamma \in H^2(\overline{M}_{n,1}(\mathbb{A}))$ we have

$$(11) \quad \begin{aligned} \phi^n(\alpha, \beta \cup \gamma) &= \sum_I n_I \phi^{n-|I|}(\mu_V^{|I|}(\alpha_i, i \in I; \cdot), \alpha_j, j \notin I; \cdot)(\iota_I^* \beta) \\ &+ \sum_{r, [I_1, \dots, I_r]} n_{[I_1, \dots, I_r]} \mu_W^s(\phi^{I_1}(\alpha_i, i \in I_1; \cdot), \dots, \phi^{I_r}(\alpha_i, i \in I_r; \cdot); \cdot)(\iota_{[I_1, \dots, I_r]}^* \beta). \end{aligned}$$

where the sum is over unordered partitions $[I_1, \dots, I_r]$ of $\{1, \dots, n\}$ with some I_j possibly empty, \cdot indicates insertion of the Künneth components of $\iota_I^* \beta$, $\iota_{[I_1, \dots, I_r]}^* \beta$, using the homeomorphisms (8), (9), the sum on the right-hand-side is assumed finite, and by convention if $n = 0$ we replace $H(\overline{M}_{n,1}(\mathbb{A}))$ with Λ (since in this case $\overline{M}_{n,1}(\mathbb{A})$ is empty). A morphism of CohFT algebras ϕ is *flat* resp. *curved* if ϕ^0 is zero resp. non-zero.

Remark 2.41. (a) In our examples, $\Lambda = \cup_{a \in \mathbb{R}} \Lambda$ will be a filtered ring, and $V = \cup_{a \in \mathbb{R}} V$, $W = \cup_{a \in \mathbb{R}} W$ filtered Λ -modules. We say that a *morphism of filtered CohFT algebras* is defined as above but where the right-hand-side of (11) is finite modulo W_a for any $a \in \mathbb{R}$.

- (b) See Nguyen-Woodward-Ziltener [70] for a description of the space of Cartier divisors in $\overline{M}_{n,1}(\mathbb{A})$, that is, a description of which combinations of codimension two strata have dual classes.
- (c) The definition of flat morphism of CohFT algebras (which has nothing to do with flat morphism of rings etc.) is analogous to the definition of flat morphism of A_∞ algebras in [26]. That is, ϕ^0 is analogous to the *curvature* of a A_∞ morphism.
- (d) The divisors $D_{\{1\}, \{2\}}, D_{\{1,2\}} \subset \overline{M}_{2,1} \cong \mathbb{P}$ are points (the limiting points in Figures 5, 6) and so have the same homology class. Using the splitting axiom (11) this implies that if ϕ^0 vanishes then ϕ^1 is a homomorphism from (V, μ_V^2) to (W, μ_W^2) . This is an analog of the fact that a flat A_∞ morphism induces an algebra homomorphism of cohomology groups.
- (e) For simplicity we will consider here only the even case, that is, V is a usual vector space and there are no signs.

Now we discuss the connection of morphisms of CohFT algebras with Frobenius manifolds, or rather, the underlying family of algebras:

Definition 2.42. Let V, W be vector spaces equipped with associative products $\star_v : T_v V^2 \rightarrow T_v V$, $\star_w : T_w W^2 \rightarrow T_w W$ varying smoothly in v, w . A \star -morphism from V to W is an analytic map $\phi : V \rightarrow W$ whose derivative $D_v \phi$ is a morphism of algebras from $T_v V$ to $T_{\phi(v)} W$ for all $v \in V$.

In particular, if $\phi : V \rightarrow W$ is a \star -morphism with Taylor coefficients ϕ^n then $D_0 \phi = \phi^1$ is an algebra homomorphism from $T_0 V$ to $T_{\phi^0(1)} W$.

Proposition 2.43. Any morphism of CohFT algebras $(\phi^n)_{n \geq 0}$ from V to W defines a formal \star -morphism from V to W via the formula

$$\phi : V \rightarrow W, \quad v \mapsto \sum_{n \geq 0} \frac{1}{n!} \phi^n(v, \dots, v).$$

ϕ arises from a flat morphism of CohFT algebras iff $\phi(0) = 0$.

Proof. For convenience, we reproduce the argument from [70, Proposition 2.43]. Consider the relation

$$[D_{\{1,2\}}] = [D_{\{1\},\{2\}}] \in H^2(\overline{M}_{2,1}(\mathbb{A}))$$

from Remark 2.41 (c). Its pull-back under the morphism $\overline{M}_{n,1}(\mathbb{A}) \rightarrow \overline{M}_{2,1}(\mathbb{A})$ forgetting all but the first two markings is the relation

$$\sum_{r, [I_1, \dots, I_r]} [D_{[I_1, I_2, \dots, I_r]}] = \sum_I [D_I]$$

where the first sum is over partitions I_1, \dots, I_r with $1 \in I_1, 2 \in I_2$, and the second is over subsets $I \subset \{1, \dots, n\}$ with $\{1, 2\} \subset I$. The splitting axiom implies for each $a, b \in T_v V$

$$\begin{aligned} D_v \phi(a \star_v b) &= \sum_{n,i} \frac{1}{(i-2)!(n-i)!} \phi^{n-i+1}(\mu_V^i(a, b, v, \dots, v; 1), v, \dots, v; 1) \\ &= \sum_{n,I} ((n-2)!)^{-1} \phi^{n-|I|+1}(\mu_V^{|I|}(a, b, v, \dots, v; 1), v, \dots, v; 1) \\ &= \sum_{I_1 \ni 1, I_2 \ni 2, I_3, \dots, I_r} ((n-2)! \# \{j \mid I_j = \emptyset\})^{-1} \mu_W^r(\phi^{|I_1|}(a, v, \dots, v; 1), \\ &\quad \phi^{|I_2|}(b, v, \dots, v; 1), \phi^{|I_3|}(v, \dots, v; 1), \dots, \phi^{|I_r|}(v, \dots, v; 1); 1) \\ &= \sum_{i_1, i_2 \geq 1, i_3, \dots, i_r \geq 0} \frac{1}{(i_1-1)!(i_2-1)!i_3! \dots i_r!(r-2)!} \mu_W^r(\phi^{i_1}(a, v, \dots, v; 1), \\ &\quad \phi^{i_2}(b, v, \dots, v; 1), \phi^{i_3}(v, \dots, v; 1), \dots, \phi^{i_r}(v, \dots, v; 1); 1) \\ &= \sum_r \frac{1}{(r-2)!} \mu_W^r(D_v \phi(a), D_v \phi(b), \phi(v), \dots, \phi(v)) \\ &= D_v \phi(a) \star_{\phi(v)} D_v \phi(b) \end{aligned}$$

where the right-hand-side is assumed to be a finite sum in each graded piece. By definition $\phi(0) = 0$ iff ϕ^0 vanishes iff $(\phi^n)_{n \geq 0}$ is flat. \square

2.4. Compositions of morphisms and traces. Morphisms and traces on a CohFT algebra admit a notion of composition, which generalizes the usual homotopy notions of composition in the A_∞ setting.

The definition of 2-morphism of a composition of a trace with a morphism depends on a moduli space of *scaled parametrized curves* which combines features of the complexified multiplihedron and cyclohedron. Let $M_{n,1}(C)$ denote the space of n -marked 1-scaled curves with underlying curve C ; we do not quotient by automorphisms of C . The space $M_{n,1}(C)$ admits a compactification $\overline{M}_{n,1}(C)$ by allowing *stable scaled curves* allowing bubbles with zero area form or allowing the area form on C to degenerate to zero and *affine scaled curves* to develop as bubbles. Recall that any nodal map $u : \hat{C} \rightarrow C$ of class $[C]$ has a *relative dualizing sheaf* given as the tensor product of the dualizing sheaf $T^\vee \hat{C}$ for \hat{C} and the inverse of the pull-back of the cotangent bundle $T^\vee C$ to C :

$$T_u^\vee := T^\vee \hat{C} \otimes (u^* T^\vee C)^{-1};$$

(More detail is given below in Example 4.2.) A section of T_u^\vee consists of a collection of rational one-forms $\lambda_i : C_i \rightarrow T^\vee C_i$ on the bubble components, and a function $\lambda_0 : C_0 \rightarrow \mathbb{C}$ on the principal component, such that λ_i have at most simple poles at the nodes and if so, the residues on either side of the node match. We denote by $\mathbb{P}(T_u^\vee \oplus \mathbb{C})$ the associated bundle with projective line fibers.

Definition 2.44. (Nodal Scaled Marked Curves) Let $u : \hat{C} \rightarrow C$ be a map of class $[C]$. A *scaling form* is a section $\lambda : \hat{C} \rightarrow \mathbb{P}(T_u^\vee \oplus \mathbb{C})$ such that on any connected component \hat{C}' of $\hat{C} - C_0$ (that is, bubble tree attached to the principal component) the pair $(\hat{C}', \lambda|_{\hat{C}'})$ is an affine scaled curve (if the scaling λ is infinite on C_0) or has zero scaling, otherwise. A *nodal scaled curve* parametrized by C is a map $\hat{C} \rightarrow C$ equipped with a scaling form. An *isomorphism* of nodal scaled parametrized curves $(\hat{C}_j, u_j, \omega_j), j = 0, 1$, is an isomorphism $\psi : \hat{C}_0 \rightarrow \hat{C}_1$ such that

- (a) (Scalings are intertwined) $\psi^* \lambda_1 = \lambda_0$.
- (b) (Markings are intertwined) $\psi(z_{0,i}) = z_{1,i}, i = 1, \dots, n$;
- (c) (Parametrization is intertwined) $\psi \circ u_0 = u_1$.

A nodal scaled parametrized curve is *stable* iff it has no infinitesimal automorphisms, that is, each irreducible non-principal component has at least three special points or a non-degenerate scaling and two special points. The *combinatorial type* of a nodal scaled parametrized curve is a colored tree $\Gamma = (\text{Vert}(\Gamma), E(\Gamma))$ with finite resp. semiinfinite edges $E_{<\infty}(\Gamma)$ resp. $E_\infty(\Gamma)$ obtained by replacing every irreducible component by a vertex and every node or marking with an edge, equipped with a *root vertex* $v_0 \in \text{Vert}(\Gamma)$ corresponding to the principal component and a partition

$$\text{Vert}(\Gamma) = \text{Vert}^0(\Gamma) \cup \text{Vert}^1(\Gamma) \cup \text{Vert}^\infty(\Gamma)$$

corresponding to the irreducible components with zero resp. finite resp. infinite scalings. A rooted colored tree is *stable* if it corresponds to a stable scaled curve, that is, every non-root vertex in $\text{Vert}^0(\Gamma)$ or $\text{Vert}^\infty(\Gamma)$ resp. $\text{Vert}^1(\Gamma)$ has at least 3 resp. 2 incident edges.

In other words, a stable scaled curve is a copy of the curve C with a section of the trivial bundle (necessarily constant) and a collection of stable curves attached (if the scaling on the principal component is zero or finite) or a curve with infinite scaling and a collection of stable scaled affine lines attached (if the scaling on the principal component is infinite).

Let $\overline{M}_{n,1}(C)$ denote the moduli space of isomorphism classes of n -marked, scaled curves with principal component C , and $M_{n,1,\Gamma}(C)$ the subset of combinatorial type Γ so that

$$\overline{M}_{n,1}(C) = \cup_{\Gamma} M_{n,1,\Gamma}(C).$$

The topology on $\overline{M}_{n,1}(C)$ is similar to that for $\overline{M}_{n,1}(\mathbb{A})$ and is compact and Hausdorff. The local structure of $\overline{M}_{n,1}(C)$ near the stratum $M_{n,1,\Gamma}(C)$ of nodal lines with combinatorial type Γ may be described as follows. As in (7), for any colored rooted tree Γ , a labelling $\gamma : \text{Edge}_{<\infty}(\Gamma) \rightarrow \mathbb{C}$ is *balanced* iff

$$(12) \quad \prod_{e \in P_{vw}} \gamma(e)^{\pm} = 1$$

where v, w range over elements of $\text{Vert}^1(\Gamma)$ and P_{vw} is the unique non self-crossing path from v to w , and the product the sign is positive if e is pointing towards the root vertex and negative otherwise. Let Z_{Γ} denote the set of balanced labellings. As in [61] for each rooted tree Γ there exists a tubular neighborhood of the form

$$M_{n,1,\Gamma}(C) \times Z_{\Gamma} \rightarrow \overline{M}_{n,1}(C)$$

given by removing small neighborhoods of the nodes and gluing together using identifications depending on the gluing parameters.

Example 2.45. Suppose that \hat{C} consists of a principal component $C_0 \cong C$ with infinite scaling, two other components with infinite scaling (dark shading), four components with finite scaling (medium shading), and two components with zero scaling (light shading) as shown in Figure 9.23. The relations on the gluing parameters $\gamma_1 = \gamma_2, \gamma_3 = \gamma_4, \gamma_1\gamma_5 = \gamma_3\gamma_6$ imply that the curve obtained with gluing with non-zero gluing parameters is equipped with the area form $\gamma_1\gamma_5\omega_C = \gamma_2\gamma_5\omega_C = \gamma_3\gamma_6\omega_C = \gamma_4\gamma_6\omega_C$.

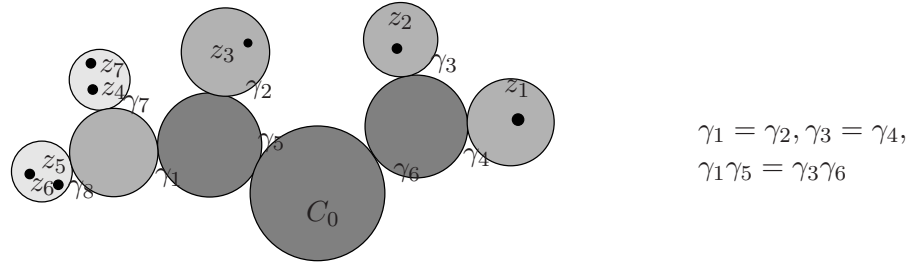


FIGURE 13. Gluing relations on a marked scaled curve

Proposition 2.46. *The boundary of $\overline{M}_{n,1}(C)$ is the union of the following sets (which will be divisors once the algebraic structure on $\overline{M}_{n,1}(C)$ is introduced)*

- (a) (Bubbling points) *For any subset $I \subset \{1, \dots, n\}$ of order at least two we have a subspace*

$$\iota_I : D_I \rightarrow \overline{M}_{n,1}(C)$$

and an isomorphism

$$\varphi_I : D_I \rightarrow \overline{M}_{0,|I|+1} \times \overline{M}_{n-|I|+1,1}(C)$$

corresponding to the formation of a bubble with markings $z_i, i \in I$ with zero scaling.

- (b) (Blowing up scaling) For any unordered partition $[I_1, \dots, I_r]$ of $\{1, \dots, n\}$ we have a subspace

$$\iota_{[I_1, \dots, I_r]} : D_{[I_1, \dots, I_r]} \rightarrow \overline{M}_{n,1}(C)$$

and an isomorphism

$$\varphi_{[I_1, \dots, I_r]} : D_{[I_1, \dots, I_r]} \rightarrow \overline{M}_r(C) \times \prod_{j=1}^r \overline{M}_{|I_j|,1}(\mathbb{A})$$

corresponding to degeneration of the area form to infinity, and bubbles with non-degenerate area form containing the markings.

- (c) (Fixing a scaling) For any $\rho \in \mathbb{C}$, particular for $\rho = 0$, there is an inclusion

$$\iota_\rho : \overline{M}_n(C) \rightarrow \overline{M}_{n,1}(C)$$

by choosing any scaling $\rho\omega_C$.

Proof. The description of boundary subspaces is immediate from the tubular neighborhood description of each stratum and a dimension count, which is the same as in Remark 2.37. \square

The adiabatic limit Theorem 1.5 will be deduced from the following divisor class relation:

Proposition 2.47. (The basic divisor class relation in the moduli of stable scaled curves) *The homology class of $\iota_\rho(\overline{M}_n(C))$ is equal to that of the union of classes of $D_{[I_1, \dots, I_r]}$ over unordered partitions.*

Proof. The equivalence in homology induced by the map $\rho : \overline{M}_{n,1}(C) \rightarrow \mathbb{P}$ equates the homology classes of $\rho^{-1}(0) \cong \overline{M}_n(C)$ with $\rho^{-1}(\infty) = \cup_{r, [I_1, \dots, I_r]} D_{[I_1, \dots, I_r]}$, as one can check that the multiplicity of each divisor on the right hand side is 1, using the fact that linear equivalence of divisors implies homology equivalence. \square

Suppose that V, W are (even, genus zero) CohFT algebras, with structure maps

$$\mu_V^n : V^n \times H(\overline{M}_{0,n+1}) \rightarrow V, \quad \mu_W^n : W^n \times H(\overline{M}_{0,n+1}) \rightarrow W.$$

Let $\phi^n : V^n \times H(\overline{M}_{n,1}(\mathbb{A})) \rightarrow V$ be a morphism of CohFT algebras, and τ_V, τ_W traces on V, W respectively.

Definition 2.48. (2-morphisms for compositions of traces and morphisms) A 2-morphism from $\phi \circ \tau_W$ to τ_V is a collection of maps

$$\psi : V^n \times H(\overline{M}_{n,1}(C)) \rightarrow W, \quad n \geq 0$$

such that

- (a) (Fixing Scaling) if $\gamma_\rho \in H^2(\overline{M}_{n,1}(C))$ is the dual class to $\iota_\rho(\overline{M}_n(C))$ then

$$\psi^n(\alpha_1, \dots, \alpha_n; \beta \cup \gamma_\rho) = \tau_V(\alpha_1, \dots, \alpha_n; \iota_\rho^* \beta);$$

- (b) (Bubbling points) if $\gamma_I \in H^2(\overline{M}_{n,1}(C))$ is the dual class to the divisor D_I corresponding to bubbling off markings $z_i, i \in I$, then

$$\psi^n(\alpha_1, \dots, \alpha_n; \beta \cup \gamma_I) = \psi^{n-|I|+1}(\alpha_j, j \notin I, \mu^{|I|}(\alpha_i, i \in I, \cdot); \cdot)(\iota_{I,1}^* \beta);$$

- (c) (Blowing up scaling) if $D = \sum n_{[I_1, \dots, I_r]} D_{[I_1, \dots, I_r]}$ is a boundary divisor with dual class γ then

$$(13) \quad \psi^n(\alpha_1, \dots, \alpha_n; \beta \cup \gamma) = \sum_{[I_1, \dots, I_r]} \tau_W^r(\phi^{|I_1|}(\alpha_i, i \in I_1; \cdot), \dots, \phi^{|I_r|}(\alpha_i, i \in I_r; \cdot); \cdot)(\iota_{[I_1, \dots, I_r]}^* \beta).$$

We write $\tau_V \cong_\psi \tau_W \circ \phi$.

Lemma 2.49. *If $\tau_V \cong_\psi \tau_W \circ \phi$ then*

$$\tau_V(\alpha_1, \dots, \alpha_n; \iota_\rho^* \beta) = \sum_{r, [I_1, \dots, I_r]} \tau_W^r(\phi^{|I_1|}(\alpha_i, i \in I_1; \cdot), \dots, \phi^{|I_r|}(\alpha_i, i \in I_r; \cdot); \cdot)(\iota_{[I_1, \dots, I_r]}^* \beta)$$

Proof. By combining (a) and (c) in the definition and using the equality of the homology class $[D_\rho]$ of the divisor D_ρ corresponding to fixed scaling and the divisor at infinite scaling $\sum [D_{[I_1, \dots, I_r]}]$. \square

Lemma 2.50. *If g_V, g_W are the metrics on CohFT algebras V, W defined by traces τ_V, τ_W and there is a 2-morphism $\tau_V \cong_\psi \tau_W \circ \phi$ then $\phi : V \rightarrow W$ is an isometry in the sense that $g_V(v_1, v_2) = g_W(D_v \phi(v_1), D_v \phi(v_2))$.*

Proof. We restrict to the case $\phi^0 = 0$ for simplicity. Since $\tau_V \cong_\psi \tau_W \circ \phi$ we have

$$\begin{aligned} g_V(v_1, v_2) &= \sum_{n \geq 0} \frac{1}{n!} \tau_V^{n+2}(v_1, v_2, v, \dots, v; \gamma_1 \cup \gamma_2) \\ &= \sum_{n \geq 0, i_1, \dots, i_r} ((i_1 - 1)!(i_2 - 1)!i_3! \dots i_r!(r - 2)!)^{-1} \tau_W^r(\phi^{i_1}(v_1, v, \dots, v), \\ &\quad \phi^{i_2}(v_2, v, \dots, v), \phi^{i_3}(v, \dots, v), \dots, \phi^{i_r}(v, \dots, v); \gamma_1 \cup \gamma_2) \\ &= g_W(D_v \phi(v_1), D_v \phi(v_2)). \end{aligned}$$

\square

We next discuss compositions of morphisms of CohFT algebras. These are not used in the paper, but play a role in the “quantum reduction in stages” Conjecture 8.10 and complete the picture of CohFT algebras as complex analogs of A_∞ algebras. The definition depends on the following moduli space, which again is a complexification of a space that was briefly mentioned in Stasheff [86].

Definition 2.51. (Multiply scaled, marked affine lines) An s -scaled, n -marked affine line is an affine line \mathbb{A} equipped with s scalings $\lambda_1, \dots, \lambda_s \in \Omega^1(\mathbb{A})^\mathbb{C} - \{0\}$ as in Definition 2.23 and an n -tuple of distinct points z_1, \dots, z_n in \mathbb{A} . An *isomorphism* of s -scaled, n -marked affine lines $(\mathbb{A}, \underline{\lambda}, \underline{z})$ to $(\mathbb{A}', \underline{\lambda}', \underline{z}')$ is an isomorphism $\psi : \mathbb{A} \rightarrow \mathbb{A}'$ preserving the scalings and markings: $\lambda_{i,0} = \psi^* \lambda_{i,1}$ for $i = 1, \dots, s$ and $\psi(z_j^0) = z_j^1$ for $j = 1, \dots, n$.

Let $M_{n,s}(\mathbb{A})$ denote the moduli space of isomorphism classes of s -scaled, n -marked affine lines. $M_{n,s}(\mathbb{A})$ has a natural compactification obtained by allowing bubbles on which a proper subset of the scalings have gone to infinity or zero.

Definition 2.52. (Multiply scaled, marked nodal affine lines) A s -scaled n -marked nodal curve $(C, \underline{z}, \underline{\lambda})$ consists of

- (a) (Curve) a nodal curve C with irreducible components denoted C_0, \dots, C_k
- (b) (Markings) A $n + 1$ -tuple of distinct smooth points $\underline{z} = (z_0, \dots, z_n) \in C^{n+1}$, with $z_0 \in C_0$;
- (c) (Scalings) a collection of possibly degenerate scalings $\underline{\lambda} = (\lambda_1, \dots, \lambda_s)$, where each $\lambda_j : C \rightarrow \mathbb{P}(T^\vee C \oplus \mathbb{C})$ is a scaling as in Definition 2.23

such that

- (a) (Monotone) for any non-self-crossing path from the root marking z_0 to a marking $z_i, i > 0$, there is exactly one irreducible component $m(i, j)$ on which the j -th scaling is non-degenerate; the irreducible components previous to $m(i, j)$ have infinite j -th scaling and the irreducible components after that have zero j -th scaling.
- (b) (Balanced) For any indices j, k , the ratio between the j -th and k -th scaling is independent of the choice of irreducible component. This means that if there is an irreducible component on which λ_j and λ_k are both non-zero and finite, then the ratio λ_j/λ_k is independent of such an irreducible component and λ_j is finite whenever λ_k is, and if there is no such irreducible component, then for each non-self-crossing path from z_0 to $z_i, i > 0$ either $m(i, j) < m(i, k)$ for all $i = 1, \dots, n$ or $m(i, j) > m(i, k)$ for all $i = 1, \dots, n$ with respect to the ordering given by the distance from root marking z_0 .

A s -scaled, n -marked line C is *stable* if it has no infinitesimal automorphisms, that is, each irreducible component of C with at least one non-degenerate scaling and two special points, or three special points. The *combinatorial type* of a nodal s -scaled, n -marked affine line $(C, \underline{\lambda}, \underline{z})$ is an s -colored tree, whose vertices correspond to irreducible components of C , whose finite edges correspond to nodes, whose semiinfinite edges correspond to markings, and for each $j = 1, \dots, s$ the set of j -colored vertices $\text{Vert}^j(\Gamma) \subset \text{Vert}(\Gamma)$ corresponds to irreducible components on which the j -scaling is non-zero and finite. The colorings satisfy the monotonicity and balanced conditions above. An s -colored tree is stable if it corresponds to a stable s -scaled affine line, that is, each vertex that is not resp. is in one of the sets $\text{Vert}^j(\Gamma)$ has valence at least three resp. two.

Let $\overline{M}_{n,s}(\mathbb{A})$ denote the set of isomorphism classes of connected s -scaled, n -marked affine lines. It has a natural topology, similar to that for $\overline{M}_{n,1}(\mathbb{A})$ and is compact and Hausdorff.

Example 2.53. $\overline{M}_{1,2}(\mathbb{A})$ is the moduli space of twice scaled affine lines with one finite marking and a marking at infinity. Using the translation to put the finite marking at zero the open stratum $M_{1,2}(\mathbb{A})$ consists of isomorphism classes of pairs $(\lambda_1 = w_1 dz, \lambda_2 = w_2 dz)$, identified with \mathbb{C}^\times by the map $[(w_1 dz, w_2 dz)] \mapsto w_2/w_1$. The two boundary points occur when $w_1/w_2 \rightarrow 0$ or $w_1/w_2 \rightarrow \infty$. In each case the limiting configuration consists of two irreducible components C_0, C_1 where some λ_i is finite on C_0 and zero on C_1 , and $\lambda_j, j \neq i$ is infinite on C_0 and finite on C_1 . See Figure 14, where the root marking z_0 is on the bottom.

Proposition 2.54. *The boundary of $\overline{M}_{n,s}(\mathbb{A})$ is the union of the following sets (which will be divisors once the algebraic structure on $\overline{M}_{n,s}(\mathbb{A})$ is introduced)*

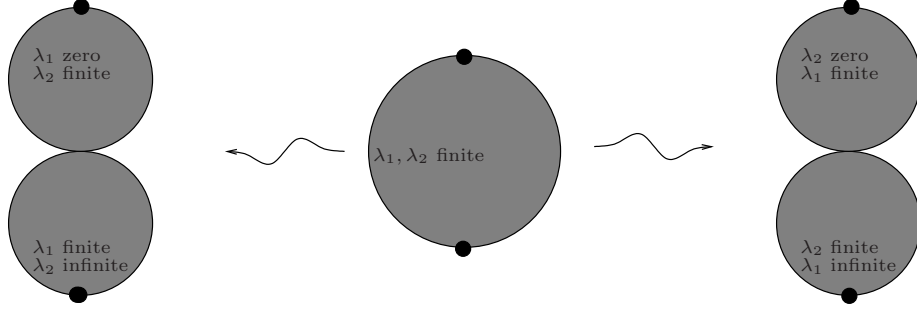


FIGURE 14. Degeneration of curves with two scalings

- (a) (Bubbling points) *For any subset $I \subset \{1, \dots, n\}$ of order at least two there is a subspace*

$$\iota_I : D_I \rightarrow \overline{M}_{n,s}(\mathbb{A})$$

and an isomorphism

$$\varphi_I : D_I \rightarrow \overline{M}_{0,|I|+1} \times \overline{M}_{n-|I|+1,s}(\mathbb{A})$$

corresponding to the formation of a bubble containing the markings $z_i, i \in I$ with zero scalings on that bubble.

- (b) (Degenerating Scalings) *For any unordered partition $[I_1, \dots, I_r]$ of $\{1, \dots, n\}$ of order at least two and subset $J \subset \{1, \dots, s\}$ there is a subspace*

$$\iota_{[I_1, \dots, I_r], J} : D_{[I_1, \dots, I_r], J} \rightarrow \overline{M}_{n,s}(\mathbb{A})$$

with an isomorphism

$$\varphi_{[I_1, \dots, I_r], J} : D_{[I_1, \dots, I_r], J} \rightarrow \overline{M}_{r+1, s-|J|}(\mathbb{A}) \times \prod_{i=1}^r \overline{M}_{|I_i|+1, |J|}(\mathbb{A})$$

corresponding to the formation of r bubbles containing markings $I_j, j = 1, \dots, r$ with the scalings $j \in J$ finite on those bubbles.

That is,

$$\partial M_{n,s}(\mathbb{A}) = \bigcup_{I \subset \{1, \dots, n\}} D_I \cup \bigcup_{J, [I_1, \dots, I_r]} D_{[I_1, \dots, I_r], J}.$$

For any unordered partition $[J_1, \dots, J_r]$ of the scaling indices $\{1, \dots, s\}$ we have a canonical embedding

$$\iota_{[J_1, \dots, J_r]} : \overline{M}_{n,r}(\mathbb{A}) \rightarrow \overline{M}_{n,s}(\mathbb{A})$$

corresponding to the locus where the scalings $j \in J_i$ are equal. In particular there is an inclusion

$$\iota_{\{1, \dots, s\}} : \overline{M}_{n,1}(\mathbb{A}) \rightarrow \overline{M}_{n,s}(\mathbb{A})$$

defined by setting all scalings equal. Consider the inclusion $\iota_{\{1, \dots, s\}}$ given by setting all scalings equal. Denote the induced map in cohomology

$$\iota_{\{1, \dots, s\}}^* : H(\overline{M}_{n,1}(\mathbb{A})) \rightarrow H(\overline{M}_{n,s}(\mathbb{A}))$$

Definition 2.55. (2-morphisms for compositions of morphisms) Given morphisms of CohFT algebras

$$(\phi_{01}^n)_{n \geq 0} : V_0 \rightarrow V_1, \quad (\phi_{12}^n)_{n \geq 0} : V_1 \rightarrow V_2, \quad (\phi_{02}^n)_{n \geq 0} : V_0 \rightarrow V_2$$

a 2-morphism from $\phi_{01} \circ \phi_{12}$ to ϕ_{02} is a collection of maps

$$\psi : V_0^n \times H(\overline{M}_{n,2}(\mathbb{A})) \rightarrow V_2, n \geq 0$$

such that $\psi(1 \times \iota_{\{1,2\}}^*) = \phi_{02}$ while

$$(14) \quad \psi(\alpha, \beta \cup \gamma_{\{1\},\{2\}}) = \sum_{r, [I_1, \dots, I_r]} \phi_{01}^r(\phi_{12}^{|I_1|}(\alpha_i, i \in I_1; \cdot), \dots, \phi_{12}^{|I_r|}(\alpha_i, i \in I_r; \cdot); \cdot)(\iota_{\{1\},\{2\}}^* \beta).$$

Here the dots indicate insertion of the Künneth components of $\iota_{\{1\},\{2\}}^*(\beta)$ with respect to the decompositions (8), (9).

We say that the diagram of CohFT algebras

$$\begin{array}{ccc} V_0 & \xrightarrow{\quad} & V_2 \\ & \searrow \quad \swarrow & \\ & V_1 & \end{array}$$

commutes. Any commutative diagram of CohFT algebras induces a corresponding commutative diagram of formal \star -morphisms $\phi_{02} = \phi_{12} \circ \phi_{01}$. There is a similar notion of *commutative simplex* of CohFT algebras. With this notion of simplex, one has a simplicial space whose vertices are CohFT algebras, edges are morphisms, triangles are 2-morphisms etc. In this way CohFT algebras form an ∞ -category.

3. SYMPLECTIC VORTICES

In physics, a *vortex* refers to a stable solution of classical field equations which has finite energy in two spacial dimensions, see for example Preskill [80] and, for a more mathematical treatment, Jaffe-Taubes [42] who classified vortices for scalar fields. In mathematics, vortices often refer to pairs of a connection and section of a line bundle satisfying an equation involving the curvature and a quadratic function of the section, see for example Bradlow [13]. Symplectic vortices are vortices in which the “field” takes values in a symplectic manifold with Hamiltonian group action. In this section we review the symplectic approach to gauged Gromov-Witten invariants, also known as symplectic vortex invariants or Hamiltonian gauged Gromov-Witten invariants, as introduced by Mundet and Salamon, see [16], [67]. If the moduli spaces of symplectic vortices are smooth, then integration over them defines the required invariants and the proofs of the Theorems 1.3, 1.5 are immediate. Of course, the moduli spaces are not smooth, or of expected dimension in general, and to define the needed virtual fundamental cycles we pass to algebraic geometry, starting in the following section.

3.1. Gauged holomorphic maps. In this section we review the construction of the moduli space of symplectic vortices as the symplectic quotient of the action of the group of gauge transformations on the space of gauged holomorphic maps; this generalizes the construction of the space of flat connections as the symplectic quotient of the action of the group of gauge transformations on the space of connections on a bundle. Unfortunately since the space of

gauged holomorphic maps is in general singular, this construction is rather formal and serves only for motivation for what follows.

Let K be a compact group with Lie algebra \mathfrak{k} . We assume that \mathfrak{k} is equipped with a K -invariant metric, inducing an identification $\mathfrak{k}^\vee \rightarrow \mathfrak{k}$. We denote by $EK \rightarrow BK$ a universal K -bundle, unique up to homotopy equivalence. For any K -space X , we denote by $X_K := X \times_K EK$ the homotopy quotient and by $H^K(X) := H(X_K)$ the equivariant cohomology.

Let $P \rightarrow C$ be a principal K -bundle over a compact surface C , and $\psi : C \rightarrow BK$ a classifying map for P . Denote by

$$\Omega(P, \mathfrak{k})^K = \{\theta \in \Omega(P, \mathfrak{k}) \mid (k^{-1})^* \theta = \text{Ad}(k) \theta\}$$

the space of K -invariant forms and by

$$\mathcal{A}(P) = \{\theta \in \Omega^1(P, \mathfrak{k})^K \mid \theta_{\pi(p)}(\xi_P(p)) = \xi, \forall \xi \in \mathfrak{k}, p \in P\}$$

the space of (principal) connections on P , where $\xi_P(p) = \frac{d}{dt}|_{t=0} p \exp(-t\xi)$ is the generating vector field at p . The space $\mathcal{A}(P)$ is an affine space with a free transitive action of the space $\Omega^1(C, P(\mathfrak{k}))$ of one-forms with values in the *adjoint bundle* $P(\mathfrak{k}) = P \times_K \mathfrak{k}$. Let

$$\mathcal{A}(P) \rightarrow \Omega^2(C, P(\mathfrak{k})), \quad A \mapsto F_A$$

denote the curvature map.

Let X be a Hamiltonian K -manifold with symplectic form ω and moment map $\Phi : X \rightarrow \mathfrak{k}^\vee$. By our convention this means that for all $\xi \in \mathfrak{k}$, we have $\omega(\xi_X, \cdot) = -d(\Phi, \xi)$ where $\xi_X(x) = \frac{d}{dt}|_{t=0} \exp(-t\xi)x$ is the generating vector field for ξ . Recall that $P(X)$ is the associated X -bundle. Sections $u : C \rightarrow P(X)$ are in one-to-one correspondence with lifts u_K of ψ to X_K . Given a section $u : C \rightarrow P(X)$, the homology class $[u]$ is defined to be the homology class $[u] := u_{K,*}[C] \in H_2^K(X, \mathbb{Z})$. The equivariant symplectic form $\omega_K \in \Omega_K^2(X)$ pulls back to $\Omega_K^2(P \times X)$ and descends to a closed, fiber-wise symplectic two-form $\omega_{P(X), A} \in \Omega^2(P(X))$ depending on the choice of connection A . Its cohomology class $[\omega_{P(X)}] \in H^2(P(X))$ is independent of the choice of connection. The *equivariant symplectic area* of u is

$$D(u) = ([u], [\omega_K]) = ([C], u^*[\omega_{P(X)}])$$

where $(\ , \)$ denotes the pairing between homology and cohomology. More concretely, we have $D(u) = \int_C u^* \omega_{P(X), A}$ independently of the connection A . We denote by $P(\Phi) : P(X) \rightarrow P(\mathfrak{k}^\vee) \cong P(\mathfrak{k})$ the map induced by Φ . A *gauged map* from C to X is a datum (P, A, u) where $A \in \mathcal{A}(P)$ and $u : C \rightarrow P(X)$ is a section. Given a metric on C , we denote by

$$* : \Omega^\bullet(C) \rightarrow \Omega^{2-\bullet}(C)$$

the associated Hodge star. The *energy* of a gauged map (A, u) is given by

$$(15) \quad E(A, u) = \frac{1}{2} \int_C * (\|d_A u\|^2 + \|F_A\|^2 + \|u^* P(\Phi)\|^2).$$

Suppose the C is a complex curve. Denote by $\mathcal{J}(X)$ the space of almost complex structures on X compatible with ω . The action of K induces an action on $\mathcal{J}(X)$, and we denote by $\mathcal{J}(X)^K$ the invariant subspace. Any connection $A \in \mathcal{A}(P)$ induces a map of spaces of almost complex structures

$$\mathcal{J}(X)^K \rightarrow \mathcal{J}(P(X)), \quad J \mapsto J_A$$

by combining the almost complex structure on X and C using the splitting defined by the connection. Let $\Gamma(C, P(X))$ denote the space of sections of $P(X)$. We denote by

$$\bar{\partial}_A : \Gamma(C, P(X)) \rightarrow \bigcup_{u \in \Gamma(C, P(X))} \Omega^{0,1}(C, u^* T^{\text{vert}} P(X))$$

the Cauchy-Riemann operator defined by J_A .

Definition 3.1. (Gauged holomorphic maps) Suppose that X is a Hamiltonian K -manifold equipped with an invariant almost complex structure J , C is a complex curve, and $P \rightarrow C$ is a principal K -bundle. A *gauged holomorphic map* for P is a pair (A, u) satisfying $\bar{\partial}_A u = 0$.

Let $\mathcal{H}(P, X)$ be the space of gauged holomorphic maps for P :

$$\mathcal{H}(P, X) = \{(A, u) \in \mathcal{A}(P) \times \Gamma(C, P(X)), \bar{\partial}_A u = 0\}.$$

The energy and symplectic area are related by a generalization of the familiar energy-area relation for pseudoholomorphic maps in [62]:

Proposition 3.2. (Energy-area relation, [16, Proposition 3.1]) *Suppose that X is a Hamiltonian K -manifold equipped with an invariant almost complex structure J , C is a compact complex curve, and $P \rightarrow C$ is a principal K -bundle. Let (A, u) be a gauged map from C to X with bundle P . Let ω_C be the area form determined by a choice of metric on C . The energy and equivariant symplectic area are related by*

$$(16) \quad E(A, u) = D(u) + \int_C * \|\bar{\partial}_A u\|^2 + * \frac{1}{2} \|F_A + u^* P(\Phi) \omega_C\|^2.$$

The second term in this equation has a symplectic interpretation. Formally the space $\mathcal{H}(P, X)$ has a closed two-form induced from the sum of the symplectic form on the affine space of connections and the space of maps to X . Let $P(\omega)$ denote the fiber-wise two-form on $P(X)$ defined by ω . Let $\omega_C \in \Omega^2(C)$ be the area form defined by the metric and consider the formal two-form on $\mathcal{H}(P, X)$

$$(17) \quad ((a_1, v_1), (a_2, v_2)) \rightarrow \int_C (a_1 \wedge a_2) + (u^* P(\omega))(v_1, v_2) \omega_C$$

where $(a_1 \wedge a_2) \in \Omega^2(C, \mathbb{R})$ denotes the real-valued two-form induced by wedge product and the metric on $P(\mathfrak{k})$. The form (17) is the formal restriction of a closed two-form on the space of all sections $C \rightarrow P(X)$, and so restricts to a closed two-form on the smooth locus of $\mathcal{H}(P, X)$. If X is Kähler, then the moduli space of holomorphic sections, where smooth, is an almost complex manifold, and this can be used to show non-degeneracy of (17). Let $\mathcal{K}(P)$ denote the group of gauge transformations, with Lie algebra the space of sections

$$\mathfrak{k}(P) = \Omega^0(C, P(\mathfrak{k}))$$

of the adjoint bundle $P(\mathfrak{k})$ with Lie bracket defined by the map $P(\mathfrak{k}) \times P(\mathfrak{k}) \rightarrow P(\mathfrak{k})$ induced by Lie bracket on \mathfrak{k} . The action of $\mathcal{K}(P)$ on $\mathcal{H}(P, X)$ has generating vector fields given by the covariant derivative and infinitesimal action

$$\xi_{\mathcal{H}(P, X)}(A, u) = (d_A \xi, \xi_X(u)) \in \Omega^1(C, P(\mathfrak{k})) \times \Omega^0(C, u^* T^{\text{vert}} P(X)), \quad \xi \in \Omega^0(C, P(\mathfrak{k}))$$

where $\xi_X \in \text{Map}(P(X), P(TX))$ is induced by the infinitesimal action $\mathfrak{k} \times X \rightarrow TX$ of K .

Proposition 3.3. *The action of $\mathcal{K}(P)$ preserves the two-form (17) and has formal moment map given by the curvature plus pull-back of the moment map for X ,*

$$\mathcal{H}(P, X) \rightarrow \Omega^2(C, P(\mathfrak{k})), \quad (A, u) \mapsto F_A + u^*P(\Phi)\omega_C.$$

Proof. The action preserves the two-form by invariance of the inner product (\cdot, \cdot) on \mathfrak{k} and symplectic form ω on X . To prove the moment map condition, note that for $(A, u) \in \mathcal{H}(P, X)$ and $(a, v) \in T_{A,u}\mathcal{H}(P, X)$ we have

$$\begin{aligned} \int_C (d_A \xi \wedge a) + P(\omega)(\xi_X(u), v) &= \int_C (-\xi \wedge d_A a) - L_v u^*(P(\Phi), \xi) \\ &= -L_{(a,v)}(F_A + u^*P(\Phi)\omega_C, \xi) \end{aligned}$$

where L_v resp. $L_{(a,v)}$ denotes the derivative in the direction of v resp. (a, v) . \square

Definition 3.4. A gauged map $(A, u) \in \mathcal{H}(P, X)$ is a *symplectic vortex* if it satisfies the

$$(\text{Vortex Equation}) \quad F_{A,u} := F_A + u^*P(\Phi)\omega_C = 0.$$

An *isomorphism* of symplectic vortices $(A_j, u_j), j = 0, 1$ with bundle P is a gauge transformation $\phi : P \rightarrow P$ such that $\phi^*A_1 = A_0$ and $[\phi, \text{Id}_X] \circ u_0 = u_1$, where $[\phi, \text{Id}_X] : P(X) \rightarrow P(X)$ is the fiber-bundle-automorphism induced by ϕ . An *n-marked* symplectic vortex is a vortex (A, u) together with an n -tuple $\underline{z} = (z_1, \dots, z_n)$ of distinct points in C . A *framed vortex* is a collection $(A, u, \underline{z}, \underline{\phi})$, where (A, u, \underline{z}) is a marked vortex and $\underline{\phi} = (\phi_1, \dots, \phi_n)$ is an n -tuple where each $\phi_j : P_{z_j} \rightarrow K$ is a K -equivariant isomorphism, that is, a trivialization of the fiber.

Let $M_n(P, X)$ denote the moduli space of isomorphism classes of n -marked vortices and $M_n^K(C, X)$ the union over isomorphism classes of bundles $P \rightarrow C$. The moduli space $M_n^K(C, X)$ is homeomorphic to the product $M^K(C, X) \times M_n(C)$ where $M_n(C)$ denotes the configuration space of n -tuples of distinct points in C .

3.2. Nodal symplectic vortices. The bubbling phenomenon for holomorphic curves also occurs in the case of symplectic vortices and prevents compactness of the moduli spaces. However, once one incorporates bubbles and fixes the homology class the moduli spaces become compact, as we now explain following Salamon et al [16] who proved compactness of the moduli space of vortices of fixed homology class in the case that X has no holomorphic spheres, and Mundet [67] and Ott [77] who compactify the moduli space of vortices by allowing bubbling in the fibers of $P(X)$.

Definition 3.5. (Nodal vortices) Suppose that X is a Hamiltonian K -manifold equipped with an invariant almost complex structure J and C is a connected smooth projective curve. A *nodal gauged n -marked map* from C to X with underlying bundle P consists of a datum $(P, A, \hat{C}, u, \underline{z})$ where

- (a) (Bundle and Connection) $P \rightarrow C$ is a K -bundle and A is a connection on P ;
- (b) (Stable section) $u : \hat{C} \rightarrow P(X)$ is a stable map of base degree one, that is, $\pi \circ u : \hat{C} \rightarrow C$ is a stable map of class $[C]$;
- (c) (Markings) an n -tuple $\underline{z} = (z_1, \dots, z_n) \in \hat{C}^n$ of distinct, smooth points of \hat{C} .

An *isomorphism* of nodal gauged maps $(\hat{C}', A', u', \underline{z}'), (\hat{C}'', A'', u'', \underline{z}'')$ with bundles P', P'' consists of

- (a) (Domain automorphism) an automorphism of the domain $\psi : \hat{C}' \rightarrow \hat{C}''$, inducing the identity on C , and
- (b) (Bundle automorphism) a bundle isomorphism $\phi : P' \rightarrow P''$, inducing the identity on C ,

intertwining the connections and maps and exchanging the markings, that is,

- (a) (Isomorphism of connections) $\phi^* A'' = A'$,
- (b) (Isomorphism of maps) $u'' = \phi(X) \circ u' \circ \psi$ where $\phi(X) : P(X) \rightarrow P'(X)$ is the map induced by ϕ and
- (c) (Isomorphism of markings) $\psi(z'_i) = z''_i, i = 1, \dots, n$.

A *nodal vortex* is a nodal gauged map such that the principal component is a vortex and there are no automorphisms with trivial bundle automorphism (gauge transformation). A nodal vortex is *stable* if it has finite automorphism group under the action of gauge transformations. That is, a nodal gauged map is a vortex if the principal component is a vortex and each sphere bubble C_i on which u_i is constant has at least three marked or singular points. Note that there is no condition on the number of special points on the principal component. In particular, constant gauged maps with no markings can be stable.

For any map $u : \hat{C} \rightarrow P(X)$, the *homology class* $[u] \in H_2^K(X, \mathbb{Z})$ of u is the push-forward of $[\hat{C}]$ under a map $u_K : \hat{C} \rightarrow X_K$ obtained from a classifying map for P .

The following extends the notion of convergence to the case of nodal marked vortices.

Definition 3.6. (Convergence of nodal vortices) Suppose that X is a Hamiltonian K -manifold equipped with an invariant almost complex structure J and C is a connected smooth projective curve. Suppose that $(P_\nu, A_\nu, \hat{C}_\nu, u_\nu, \underline{z}_\nu)$ is a sequence of marked nodal vortices on C with values in X , and $(P, A, \hat{C}, u, \underline{z})$ is a nodal vortex with values in X . We say that $[(P_\nu, A_\nu, \hat{C}_\nu, u_\nu, \underline{z}_\nu)]$ *converges* to $[(P, A, \hat{C}, u, \underline{z})]$ if after a sequence of bundle isomorphism $\phi_\nu : P_\nu \rightarrow P$ the connections A_ν converge to A_∞ in the C^0 topology and $[(\hat{C}_\nu, u_\nu, \underline{z}_\nu)]$ Gromov converges to $[(\hat{C}, u, \underline{z})]$.

See Ott [77] for more details on the definition of convergence. The definition of convergence implies in particular that the curvature F_{A_ν} converges to F_{A_∞} in L^2 , but A_ν is not required to (and does not) converge to A uniformly in all derivatives. Recall from [16] a condition which guarantees compactness of moduli spaces of symplectic vortices with non-compact target:

Definition 3.7. A Hamiltonian K -manifold X with moment map $\Phi : X \rightarrow \mathfrak{k}^\vee$ is *convex at infinity* if there exists $f \in C^\infty(X)^G$ such that

$$(18) \quad (\text{Convexity condition}) \quad \langle \nabla_v \nabla f(x), v \rangle \geq 0, \quad df(x) J_x \Phi(x)^\#(x) \geq 0$$

for every $x \in X$ and $v \in T_x X$ outside of a compact subset of X .

For example, if X is a vector space with a linear Hamiltonian action of K with *proper* moment map Φ then X is convex. The following is proved by Mundet [67] and Ott [77].

Theorem 3.8. (Sequential compactness for vortices with compact domain) *Suppose that X is Hamiltonian K -manifold equipped with a proper moment map convex at infinity and an invariant almost complex structure J , C is a connected smooth compact complex curve. Any sequence of nodal symplectic vortices with bounded energy has a convergent subsequence.*

Convergence for sequences with bounded first derivative is proved as follows: Suppose that (A_ν, u_ν) is a sequence of symplectic vortices on a bundle P with smooth domain with the property that $c_\nu := \sup \|d_{A_\nu} u_\nu\| = \|d_{A_\nu} u_\nu(z_\nu)\|$ is bounded. By Uhlenbeck compactness, after gauge transformation, we may assume that A_ν converges weakly in the Sobolev $W^{1,p}$ topology and strongly in C^0 to a limit A_∞ . After putting A_ν in Coulomb gauge with respect to A_∞ , elliptic regularity for vortices (see [16, p.20]) implies that (u_ν, A_ν) has a C^l -convergent subsequence of any l .

More generally in the case of unbounded first derivative one has a bubbling analysis similar to that for pseudoholomorphic curves [16], [77]: If $\|d_{A_\nu} u_\nu(z_\nu)\| \rightarrow \infty$ and $z_\nu \rightarrow z$ we say that z is a *singular point* for the sequence (A_ν, u_ν) .

Theorem 3.9. (Bubbling analysis for vortices with compact domain) *Let X be a Hamiltonian K -manifold with proper moment map convex at infinity.*

- (a) (Removal of Singularities) [77, Theorem 1.1] *Any finite energy vortex (A, u) on the punctured disk $D - \{0\}$ extends to a vortex on D .*
- (b) (Energy Quantization) [77, Lemma 4.2] *There exists an $E_0 > 0$ such that any for any singular point of a sequence (A_ν, u_ν) , $\lim_{\epsilon \rightarrow 0} \lim_{\nu \rightarrow \infty} E(A_\nu, u_\nu | B_\epsilon(z)) > E_0$.*
- (c) (Annulus Lemma) [77, Lemma 5.9] *There exists a constant $\epsilon > 0$ such that if (A, u) is a vortex on $S^1 \times [-R, R] \rightarrow X$ then for every $\mu < 1$ there exist constants $R_0, \delta_0, c > 0$ such that with $E(A, u) < \delta_0$, then the restriction of (A, u) to $S^1 \times [-R+1, R-1]$ satisfies*

$$E((A, u)|_{[-R+s, R-s] \times S^1}) < C e^{-2\mu s} E(A, u), \quad \forall s \in [-R+1, R-1].$$

- (d) (Mean Value Inequality) [77, Corollary 2.2] *There exist constants $C, \delta, R > 0$ such that if (A, u) is a vortex on $B_r(0)$ with $E(A, u|_{B_r(0)}) < \delta$ then*

$$(1/2)\|d_A u(0)\|^2 + \|\Phi(u(0))\|^2 < (C/r^2)E(A, u|_{B_r(0)}).$$

Remark 3.10. Given these ingredients the proof of compactness goes as follows. For each bubbling sequence z_ν with $\|d_{A_\nu} u_\nu(z_\nu)\| \rightarrow \infty$ one constructs by *soft rescaling* a sequence of $J_{A_\nu^1}$ -holomorphic maps u_ν^1 on balls of increasing radius, with the property that the limit A_ν^1 is zero and u_ν^1 is an ordinary holomorphic map from \mathbb{C} to X . Note that one does not have C^2 convergence of the sequence $J_{A_\nu^1}$; however, [77, Appendix] shows that C^0 convergence of a sequence of almost complex structures is sufficient as long as one has a version of the mean value inequality, which in this case follows from the vortex equations. The limiting configuration is then constructed by induction.

Denote by $\overline{M}_n^K(C, X)$ the set of isomorphism classes of nodal vortices with connected domain. The *combinatorial type* of a nodal vortex is a rooted tree Γ with root vertex corresponding to the principal component, equipped with a labelling of vertices by elements of $H_2^K(X, \mathbb{Z})$. For any tree Γ and homology class $d \in H_2^K(X, \mathbb{Z})$ we denote by $M_{n,\Gamma}^K(C, X, d)$ the space of isomorphism classes of vortices of combinatorial type Γ , so that

$$\overline{M}_n^K(C, X) = \bigcup_{\Gamma} M_{n,\Gamma}^K(C, X, d)$$

as Γ ranges over connected combinatorial types. We say that a subset S of $\overline{M}_n^K(C, X)$ is *closed* if any convergent sequence in S has limit point in S , and *open* if its complement is closed. The

open sets form a topology for which any convergent sequence is convergent, and any convergent sequence has a unique limit, by arguments similar to [62, Lemma 5.6.5]. Namely local “distance functions” can be defined by combining the distance functions of [62] with the L^2 -metric on the space of connections. In the case $n = 0$, given a constant $\epsilon > 0$ and stable vortex (P, A_0, \hat{C}_0, u_0) and another stable vortex (P, A_1, \hat{C}_1, u_1) with the same underlying bundle P define

$$(19) \quad \text{dist}_\epsilon([(P, A_0, \hat{C}_0, u_0)], [(P, A_1, \hat{C}_1, u_1)]) = \inf_{k \in \mathcal{K}(P)} \|k \cdot A_1 - A_0\|_{L^2} + \text{dist}_\epsilon^0([\hat{C}_0, u_0], [(\hat{C}_1, k \cdot u_1)])$$

where dist_ϵ^0 is the distance on isomorphism classes of stable maps defined on [62, p. 134], but using the Yang-Mills-Higgs energy on a small ball around each node. It follows from the results in Ott [77] and elliptic regularity for vortices that for ϵ sufficiently small, a sequence $[(P, A_\nu, \hat{C}_\nu, u_\nu)]$ converges to $[(P, A_0, \hat{C}_0, u_0)]$ if and only if

$$\text{dist}_\epsilon([(P, A_0, \hat{C}_0, u_0)], [(P, A_\nu, \hat{C}_\nu, u_\nu)]) \rightarrow 0$$

and the remainder of the proof is similar to that in [62]. The sequential compactness Theorem 3.8 implies:

Theorem 3.11. (Properness of the moduli space of symplectic vortices) *Suppose that X is a Hamiltonian K -manifold equipped with a proper moment map convex at infinity and an invariant almost complex structure J and C is a connected smooth projective curve. Then $\overline{M}_n^K(C, X)$ is Hausdorff and the energy map $E : \overline{M}_n^K(C, X) \rightarrow [0, \infty)$ is proper.*

3.3. Large area limit. In the large area limit the vortices are related to holomorphic maps to the (possibly orbifold) symplectic quotient, as pointed out by Gaio-Salamon [29]. The limiting process involves various kinds of bubbling which one hopes to incorporate into a description of the relationship.

First recall the notion of symplectic quotient of X by K as introduced by Mayer and Marsden-Weinstein: Suppose that X is a Hamiltonian K -manifold equipped with a proper moment map $\Phi : X \rightarrow \mathfrak{k}^\vee$. Let

$$X//K := \Phi^{-1}(0)/K$$

denote the symplectic quotient. If K acts freely and properly on $\Phi^{-1}(0)$, then $X//K$ has the structure of a smooth manifold of dimension $\dim(X) - 2 \dim(K)$ with a unique symplectic form ω_0 satisfying $i^*\omega = p^*\omega_0$, where $i : \Phi^{-1}(0) \rightarrow X$ and $p : \Phi^{-1}(0) \rightarrow X//K$ are the inclusion and projection respectively. Any invariant almost complex structure J on X induces an almost complex structure on $X//K$.

If $X \subset \mathbb{P}(V)$ is a smooth projectively embedded variety in a G -representation V then $X//K$ is canonically homeomorphic to the geometric invariant theory quotient $X//G$ introduced by Mumford, by a theorem of Kempf-Ness. The latter is defined as the quotient of the *semistable locus*

$$X^{\text{ss}} = \{x \in X \mid \exists k \in \mathbb{Z}_+, s \in H^0(X, \mathcal{O}_X(k))^G, s(x) \neq 0\}$$

where $\mathcal{O}_X(k)$ is the k -th tensor product of the hyperplane bundle on X . The git quotient $X//G$ is the quotient of X^{ss} by the *orbit-equivalence relation*

$$x_1 \sim x_2 \iff \overline{Gx_1} \cap \overline{Gx_2} \cap X^{\text{ss}} \neq \emptyset.$$

A point $x \in X$ is *stable* if Gx is closed in X^{ss} and the stabilizer G_x is finite. If stable=semistable in X then the points of $X//G$ are the orbits of G in X^{ss} , that is, two orbits are equivalent iff they are equal.

Let C be a connected smooth projective curve, and suppose that $X//K$ is a locally free quotient.

Definition 3.12. A gauged holomorphic map (A, u) is an *infinite-area vortex* if it satisfies $u^*P(\Phi) = 0$.

Let $M^K(C, X)_\infty$ denote the set of gauge-equivalence classes of infinite-area vortices, and $M(C, X//K)$ the set of holomorphic maps from C to $X//K$.

Proposition 3.13. *Suppose that K acts locally freely on $\Phi^{-1}(0)$. Then there is a bijection from $M^K(C, X)_\infty$ to $M(C, X//K)$.*

Proof. See Gaio-Salamon [29, Section 2]. Given an infinite-area vortex (A, u) , let $\bar{u} : C \rightarrow X//K$ denote the composition of u with the quotient map $\Phi^{-1}(0) \rightarrow X//K$. The equation $\bar{\partial}_A u = 0$ implies $\bar{\partial} \bar{u} = 0$ (since $TX \rightarrow T(X//K)$ is holomorphic) and so $\bar{u} \in M(C, X//K)$. Conversely, given $\bar{u} : C \rightarrow X//K$ let P be the pull-back of $\Phi^{-1}(0) \rightarrow X//K$, equipped with the connection given by $JTX|_{\Phi^{-1}(0)} \cap TX|_{\Phi^{-1}(0)} \cong \pi^*T(X//K)$. The equivariant map $P \rightarrow \Phi^{-1}(0)$ defines a section $u : C \rightarrow P \times_K X$. The equation $\bar{\partial} \bar{u} = 0$ implies $\bar{\partial}_A u = 0$, since J_A agrees with $J_{X//K}$ on $\pi^*T(X//K)$. If (A, u) is constructed in this way from \bar{u} then the corresponding map to $X//K$ is \bar{u} . To see that the map $(A, u) \mapsto \bar{u}$ is injective, suppose that (A, u) and (A', u') define the same holomorphic map to $X//K$. Then after gauge transformation $u = u'$ and the equations $\bar{\partial}_A u = 0 = \bar{\partial}_{A'} u'$ and local freeness of the action imply that $A = A'$. \square

Gaio-Salamon [29] studied under what conditions a sequence of symplectic vortices defined with respect to an area form $\rho_\nu \omega_C$ with $\rho_\nu \rightarrow \infty$ converge to a solution of the limiting equations:

Proposition 3.14. (Sequential compactness for bounded first derivative) *Suppose that X is a Hamiltonian K -manifold with proper moment map convex at infinity equipped with an invariant almost complex structure J and C is a connected smooth projective curve. Suppose that (A_ν, u_ν) is a sequence of symplectic vortices for $\rho_\nu \omega_C$ with the property that $\sup \|d_{A_\nu} u_\nu\|$ is bounded. Then there exists a subsequence and a sequence of gauge transformations k_ν such that $k_\nu(A_\nu, u_\nu)$ converges to an infinite-area vortex (A_∞, u_∞) uniformly in all derivatives.*

In general, however, various kinds of bubbling occur.

Proposition 3.15. (Bubble zoology for the infinite area limit) *Suppose that X is a Hamiltonian K -manifold with a proper moment map convex at infinity and an invariant almost complex structure J and C is a connected smooth projective curve. Suppose that (A_ν, u_ν) is a sequence of vortices for $\rho_\nu \omega_C$, $\rho_\nu \rightarrow \infty$ with*

$$c_\nu = \sup \|d_{A_\nu} u_\nu\| = \|d_{A_\nu} u_\nu(z_\nu)\|, \quad \epsilon_\nu := \frac{\rho_\nu}{c_\nu}.$$

Consider the rescaled pair $\phi_\nu^(A_\nu, u_\nu)$ on $B_{c_\nu}(0)$ where $\phi_\nu(z) = z_\nu + z/c_\nu$. Noting that $\phi_\nu^*(A_\nu, u_\nu)$ has $\sup \|d_{\phi_\nu^* A_\nu} u_\nu\| = 1$ bounded. Then after passing to a subsequence one of the following possibilities occurs:*

- (a) (Sphere Bubble in $X//K$) If $\lim_{\nu \rightarrow \infty} \epsilon_\nu = \infty$, then $\phi_\nu^*(A_\nu, u_\nu)$ converges to a solution to 3.12, that is, is equivalent to a holomorphic map to $\mathbb{C} \rightarrow X//K$.
- (b) (Affine vortex) If $\lim_{\nu \rightarrow \infty} \epsilon_\nu \in (0, \infty)$, then $\phi_\nu^*(A_\nu, u_\nu)$ converges to a vortex on the affine line \mathbb{A} , with respect to the Euclidean area form $\omega_{\mathbb{A}} = \frac{i}{2} dz \wedge d\bar{z}$.
- (c) (Sphere Bubble in X) If $\lim_{\nu \rightarrow \infty} \epsilon_\nu = 0$, then $\phi_\nu^*(A_\nu, u_\nu)$ converges to a vortex with zero vortex parameter, that is, a holomorphic map $\mathbb{C} \rightarrow X$.

The bubble trees that occur are described further in the following section.

3.4. Affine symplectic vortices. In this section we further study vortices on the complex affine line \mathbb{A} , which arose in the Gaio-Salamon study [29] of the large area limit. Ziltener [98] studied the bubbles that arise in more detail.

Definition 3.16. (Affine vortices) Suppose that X is a compact Hamiltonian K -manifold equipped with an invariant almost complex structure J . An n -marked affine symplectic vortex with target X is a datum (A, u, \underline{z}) , where $A \in \Omega^1(\mathbb{A}, \mathfrak{g})$ is a connection on the trivial bundle $P := \mathbb{A} \times K \rightarrow \mathbb{A}$, $u : \mathbb{A} \rightarrow X$ is a J_A -holomorphic map, $\underline{z} = (z_1, \dots, z_n) \in \mathbb{A}^n$ is an n -tuple of distinct points, and

$$(\text{Affine Vortex Equation}) \quad F_A + u^* \Phi \omega_{\mathbb{A}} = 0.$$

An isomorphism of scaled vortices (A_0, u_0) to (A_1, u_1) is an automorphism ϕ of $(\mathbb{A}, \omega_{\mathbb{A}})$ (necessarily a translation) and a gauge transformation $k : \mathbb{A} \rightarrow K$ satisfying $k^* \phi^*(A_1, u_1) = (A_0, u_0)$.

Let $M_{n,1}^K(\mathbb{A}, X)$ denote the moduli space of isomorphism classes of n -marked affine vortices. It has a natural compactification by isomorphism classes of *nodal affine vortices* described as follows. Let C be an $n+1$ -marked connected genus zero nodal curve with irreducible components C_1, \dots, C_k . For each irreducible component C_i there is a unique node \hat{w}_i which disconnects C_i from the marking z_0 ; we denote by $C_i^\circ = C_i - \{\hat{w}_i\}$ the complement. Each affine curve C_i° is isomorphic to \mathbb{C} , uniquely up to translation and dilation. Thus C_i° admits a unique equivalence class of Kähler forms, equal to $\omega_{\mathbb{A}} = dz \wedge d\bar{z}(i/2)$ up to scalar multiplication.

Definition 3.17. (Nodal affine vortices) Suppose that X is a compact Hamiltonian K -manifold equipped with an invariant almost complex structure J . An n -marked nodal affine symplectic vortex with target X is a datum $(C, P, A, u, \omega, \underline{z})$ consisting of a connected $n+1$ -marked nodal curve C together with a principal K -bundle $P \rightarrow C$, a (possibly infinite or zero) two form $\omega : C \rightarrow \mathbb{P}(\Lambda^2 T_{\mathbb{R}}^\vee C \oplus \mathbb{R})$, a connection A_i on each $P|_{C_i}$, a section $u : C \rightarrow P(X)$, and markings $\underline{z} = (z_1, \dots, z_n)$, such that each irreducible component C_i of C is one of the following three types:

- (a) (Zero Scaling) Components with zero two-form, equipped with a trivial bundle $P|_{C_i}$ and a pseudoholomorphic map $u_i : C_i \rightarrow X$.
- (b) (Finite, non-zero scaling) Components with non-zero, finite area form $\omega|_{C_i} \in \Omega^2(C_i^\circ)$ equal to a non-zero multiple of $\omega_{\mathbb{A}}$ on $C_i^\circ \cong \mathbb{A}$ and an affine vortex (A_i, u_i) on C_i° .
- (c) (Infinite scaling) Components C_i with infinite two-form $\omega|_{C_i}$, equipped with holomorphic sections $u|_{C_i} : C_i \rightarrow P(X)$ mapping to the zero level set $P(\Phi^{-1}(0))$, and so defining a holomorphic map $\bar{u}_i : C_i \rightarrow X//K$.

This datum should satisfy the following conditions

- (a) (Monotonicity) For every non-self-crossing path from a marking $z_i, i > 0$ to a marking z_0 , the path crosses exactly one irreducible component with finite, non-zero area form, and all irreducible components before resp. after that irreducible component have zero resp. infinite area form.
- (b) (Continuity) If C_i meets C_j at a node represented by a pair $(w_{ij}, w_{ji}) \in C_i \times C_j$ then $u_i(w_{ij}) = u_j(w_{ji})$.
- (c) (Stability) If C_i is an irreducible component on which the two-form is zero or infinity resp. finite and non-zero and u_i is constant resp. u_i is covariant constant and A_i is flat then C_i contains at least two resp. three special points.

Remark 3.18. An irreducible component C_i is a *ghost component* if it satisfies one of the hypotheses requiring at least three special points or non-degenerate scalings; that is, \bar{u}_i is constant or $A|C_i$ is flat and $u|C_i$ is covariant constant. The stability condition can then be reformulated as the condition that any ghost component has at least three special points (nodes or markings) or two special points and a non-zero, finite area form. Either of these conditions is equivalent to the absence of non-trivial infinitesimal automorphisms: infinitesimal automorphisms arising from gauge transformations are impossible because of the local freeness assumption for the action of K on $\Phi^{-1}(0)$.

There is a natural notion of *convergence* of affine nodal vortices which generalizes convergence with fixed area form in Definition 3.6.

Definition 3.19. (Convergence of nodal affine vortices) A sequence of isomorphism classes of nodal marked affine symplectic vortices $[(C_\nu, P_\nu, A_\nu, u_\nu, \omega_\nu, \underline{z}_\nu)]$ *converges* to a nodal marked affine vortex $[(C, P, A, u, \omega, \underline{z})]$ iff there exists for each irreducible component C_j of C , a sequence $U_{j,\nu} \subset C_j$ of increasing open neighborhoods and for each ν , a holomorphic embedding $\phi_{j,\nu} : U_{j,\nu} \rightarrow C_\nu$, an isomorphism $\psi_{j,\nu} : \phi_{j,\nu}^* P_\nu \rightarrow P$ such that if $\psi_{j,\nu}(X) : \phi_{j,\nu}^* P_\nu(X) \rightarrow P(X)$ denotes the associated maps of fiber bundles then

- (a) (Open neighborhoods cover) the union of $\phi_{j,\nu}(U_{j,\nu})$ is C_ν ;
- (b) (Marked curves converge) If $z_i \in C_j$ then the limit of $\phi_{j,\nu}^{-1}(z_{\nu,i})$ is defined and equal to z_i . Furthermore, if $i \neq j$ then the image of $\phi_{j,\nu}^{-1} \circ \phi_{i,\nu}$ converges to the node of C_i connecting to C_j ;
- (c) (Connections converge) $\psi_{j,\nu}^* A_\nu$ converges to $A|C_j$ uniformly in all derivatives in any compact subset of any $U_{j,\nu'}$;
- (d) (Sections converge) $\psi_{j,\nu}(X)^* u_\nu$ converges to $u|C_j$ uniformly in all derivatives in any compact subset of any $U_{j,\nu}$;
- (e) (Scalings converge) $\phi_{j,\nu}^* \omega_\nu$ converges to $\omega|C_j$ in C^0 on any compact subset of any $U_{j,\nu'}$;
- (f) (Energies converge) $\lim_{\nu \rightarrow \infty} E(C_\nu, P_\nu, A_\nu, u_\nu, \omega_\nu, \underline{z}_\nu) = E(C, P, A, u, \omega, \underline{z})$.

Proposition 3.20. (Sequential compactness for affine vortices) *Suppose that X is a Hamiltonian K -manifold with a proper moment map convex at infinity and an invariant almost complex structure J . Any sequence of isomorphism classes of stable marked affine vortices with bounded energy has a convergent subsequence.*

The proof of the theorem above combines results of Ziltener [98], [100] who discusses bubbling of affine vortices and bubbles in $X//K$, and Ott [77], who treats bubbling in X . Namely:

Proposition 3.21. (Bubbling analysis for affine vortices) *Suppose that X is a Hamiltonian K -manifold with proper moment map convex at infinity equipped with an invariant almost complex structure J .*

- (a) (Energy quantization) [98, Lemma D.1] *There exists a constant $E_0 > 0$ such that any non-trivial symplectic vortex (A, u) on \mathbb{C} satisfies $E(A, u) > E_0$.*
- (b) (Annulus Lemma) [98, Lemma 4.11] *For every compact subset $X_0 \subset X$ and every number $r_0 > 0$ there are constants $E_1 > 0, a > 0$ and $c_1 > 0$ such that the following holds. Assume that $r_0 \leq r < R \leq \infty$ and (A, u) is a vortex on the annulus $A(r, R) = B_R(0) - B_r(0)$ such that $u(z) \in X_0$ for every $z \in A(r, R)$, and suppose that $E((A, u)|A(r, R)) \leq E_1$. Then for every $\lambda \geq 2$ we have*

$$E((A, u)|_{A(\lambda, \lambda^{-1}R)}) \leq c_1 E(A, u) \lambda^{-a}.$$

- (c) (Mean value inequality) *Let $X_0 \subset X$ be a compact subset. Then there exists a constant $E_0 > 0$ such that for every $z_0 \in \mathbb{C}$, $r > 0$ and every symplectic vortex (A, u) satisfying $u(B_r(z_0)) \subset X_0$ and $E((A, u)|_{B_r(z_0)}) \leq E_0$, the energy density $e_{A, u}$ given by the integrand in (15) satisfies the estimate*

$$e_{A, u}(z_0) := \frac{1}{2} \|d_A u(z_0)\|^2 + \|\Phi(u(z_0))\|^2 \leq (8/\pi r^2) E((A, u)|_{B_r(z_0)}).$$

- (d) (Removal of singularities) [98, Proposition D.6] *Let (A, u) be a finite energy vortex on \mathbb{C} . The map $Ku : \mathbb{C} \rightarrow X/K$ extends continuously to a map $\mathbb{P} \rightarrow X/K$, such that $Ku(\infty) \in X//K$. Furthermore,*
 - (i) *there are constants $E > 0, C > 0$ and $\delta > 0$ such that the following holds. For every vortex (A, u) on \mathbb{C} and every $R \geq 1$ such that $E(w, \mathbb{C} \setminus B_R) < E$ and every $z \in \mathbb{C} \setminus B_{2R}$ we have $e_{A, u}(z) \leq CR^\delta |z|^{-2-\delta}$;*
 - (ii) *there exist a number $\delta > 0$ such that for $2 \leq p < 4/(2 - \delta)$, then*

$$x_0 := \lim_{r \rightarrow \infty} u(r, 0)$$

exists, and there exists a map $k_0 \in W^{1,p}([0, 2\pi], G)$ such that if $A_\theta(r)$ denotes the restriction of the connection A in radial gauge to the circle $\{|z| = e^r\} \cong S^1$, then

$$\lim_{r \rightarrow \infty} \max_{\theta \in S^1} d(u(re^{i\theta}), k_0(\theta)x_0) = 0.$$

$$\sup_{r \geq 0} \|\partial_\theta k_0 k_0^{-1} + A_\theta(r)\|_{L^p(S^1)} e^{(-1+2/p+\delta/2)r} < \infty.$$

Necessarily x_0 is fixed by $k_{2\pi}$, which since K is compact and acts locally freely on $\Phi^{-1}(0)$, is finite order.

In the case that K acts freely on $\Phi^{-1}(0)$, any finite energy vortex (A, u) on \mathbb{C} has a well-defined *evaluation at infinity* $\text{ev}_\infty(A, u) \in X//K$, given by the limit of $u(s + it)$ along any ray $s + it = re^{i\theta}, r \rightarrow \infty$. More generally in the locally free case, $\text{ev}_\infty(A, u) = [k_0(2\pi), x_0]$ lies in the *inertia orbifold*

$$I_{X//K} \in \{(k, x) \in K \times \Phi^{-1}(0) \mid kx = x\}/K$$

which, for example, appeared in Kawasaki [44]. The orbifold case was not treated in Ziltener [98]; however, the proof is almost the same as the manifold case.

Remark 3.22. (Failure of removal of singularities) The area form $\omega_{\mathbb{A}}$ has a pole of order two at infinity and so one cannot expect an extension of (A, u) to the projective line to satisfy the vortex equations. Instead, the last item implies that the connection A converges in L^p , after gauge transformation, to a connection of the form $\lambda d\theta$ near infinity for some $\lambda \in \mathfrak{k}$ with $\exp(\lambda)$ finite order. By the implicit function theorem on $\mathbb{C} - B_R$ using weighted Sobolev spaces, one sees that any connection with exponential decay to $\lambda d\theta$ is complex gauge equivalent to $\lambda d\theta$, for R sufficiently large. Thus, after *complex* gauge transformation, we may take A to be equal to $\lambda d\theta$ in a neighborhood of $\infty \in \mathbb{P}$. Taking P to be a orbi-bundle on \mathbb{P}^1 given by gluing in the trivial bundle using transition map z^λ , it follows that (A, u) extends to a gauged holomorphic map on \mathbb{P} mapping ∞ to $\Phi^{-1}(0)$. However, the extension will not satisfy any vortex equation on the entire projective line \mathbb{P} .

Let $\overline{M}_{n,1}^K(\mathbb{A}, X)$ resp. $\overline{M}_{n,1}^{K,\text{fr}}(\mathbb{A}, X)$ denote the moduli space of isomorphism classes of nodal scaled affine vortices to X , resp. the moduli space of isomorphism classes of framed nodal scaled affine vortices to X . $\overline{M}_{n,1}^{\text{fr}}(\mathbb{A}, X)$ admits an evaluation maps at the markings, and, as explained in [98] an additional evaluation map at infinity to $I_{X//K}$:

$$\text{ev}^{\text{fr}} \times \text{ev}_\infty : \overline{M}_{n,1}^{K,\text{fr}}(\mathbb{A}, X) \rightarrow X^n \times I_{X//K}.$$

If K^n acts freely, combining this map with a classifying map gives a map

$$\text{ev} \times \text{ev}_\infty : \overline{M}_{n,1}^K(\mathbb{A}, X) \rightarrow X_K^n \times I_{X//K}.$$

If K^n only acts locally freely, then the map above exists as a morphism of stacks, or after passing to a classifying space for the groupoid $\overline{M}_{n,1}^K(\mathbb{A}, X)$, which has the same rational cohomology as $\overline{M}_{n,1}^K(\mathbb{A}, X)$.

Remark 3.23. The case that X and K are trivial gives the complexified multiplihedron $\overline{M}_{n,1}$ constructed in Section 2.3. Indeed, each irreducible component C_i with finite or non-zero scaling is equipped with a isomorphism with the affine line, unique up to translation, and therefore a scaling λ_i . Thus the underlying curve of any affine vortex is automatically a (possibly unstable) scaled affine curve in the sense of Section 2.3.

Let $M_{n,1}^K(\mathbb{A}, X)$ denote the moduli space of isomorphism classes of finite energy n -marked vortices on \mathbb{A} (the additional marking at infinity) with values in X . By combining the sequential compactness theorem 3.20 with local distance functions as in (19), one has:

Theorem 3.24. (Properness of the moduli space of affine vortices) *Suppose that X is a Hamiltonian K -manifold with proper moment map convex at infinity, equipped with an invariant compatible almost complex structure. $\overline{M}_{n,1}^K(\mathbb{A}, X)$ is Hausdorff and the energy map $E : \overline{M}_{n,1}^K(\mathbb{A}, X) \rightarrow [0, \infty)$ is proper.*

3.5. Stable maps to orbifolds. In order to understand bubbling in the infinite area limit we briefly review the definition of a stable maps for orbifold targets, studied in Chen-Ruan [14], which appear as bubbles in the definition of certain gauged maps. The corresponding algebraic theory by Abramovich-Graber-Vistoli [1] will be reviewed in Section 5. Recall that

Definition 3.25. (Stable pseudoholomorphic maps to symplectic orbifolds)

- (a) an *orbifold structure* on a topological space C is a proper étale groupoid together with a homeomorphism of the space of objects to C ;
- (b) An *nodal orbifold structure* on a nodal complex curve C is an orbifold structure on its normalization, such that the automorphism group of any nodal point is independent of the choice of irreducible component containing it.
- (c) A nodal orbifold structure is a *twisting* if each point with non-trivial automorphism is either a node or marking, and for each node, the orbifold charts satisfy the following *balanced condition*: the chart on one side of the node is of the form U/μ_r for some r where μ_r acts on $U \subset C$ a neighborhood of 0, while the orbifold structure on the other side is \mathbb{C}/μ_r with the conjugate action by $\exp(-2\pi i/r)$.
- (d) Let Y be a compact symplectic orbifold equipped with a compatible almost complex structure J . A *stable map* to Y is a complex nodal curve C equipped with a twisting and a representable pseudoholomorphic orbifold map $u : C \rightarrow Y$.

Remark 3.26. Representability means that the map u is smooth after passing to étale cover in Y : If $z \in C$ and $u(z) \in Y$ has automorphism group $\text{Aut}(u(z))$, then there exists an orbifold chart $U/\text{Aut}(u(z))$ for Y near $u(z)$ so that the groupoid fiber product $u^{-1}(U/\text{Aut}(u(z))) \times_{U/\text{Aut}(u(z))} U$ is an orbifold chart for z and u has a smooth local lift \tilde{u} to U given by projection on the second factor. In particular, $\text{Aut}(z)$ injects into $\text{Aut}(u(z))$.

The notion of Gromov convergence of twisted pseudoholomorphic maps which generalizes Gromov convergence of stable pseudoholomorphic maps. Let $\overline{M}_{g,n}(Y)$ denote the space of isomorphism classes of connected, genus g , n -marked stable maps to Y .

Theorem 3.27. (Properness of moduli spaces of stable maps to orbifolds) *Let Y be a compact symplectic orbifold equipped with a compatible almost complex structure. $\overline{M}_{g,n}(Y)$ is Hausdorff and the energy map $E : \overline{M}_{g,n}(Y) \rightarrow [0, \infty)$ is proper.*

Proof. By Chen-Ruan [14], who list the necessary changes in the proof of the compactness of the moduli space of stable curves for manifold targets. \square

3.6. Vortices with varying scaling. The adiabatic limit Theorem 1.5 will be proved by studying a moduli space of curves with *varying scaling* which interpolates between the moduli space of vortices for a fixed area form ω_C and the infinite area limit. More precisely, Theorem 1.5 will follow from a divisor class relation in the source moduli space constructed in Section 2.4, just as associativity of the quantum product follows from a divisor class relation in moduli space of stable curves.

First we construct the objects that appear in the infinite area limit. Let X be a Hamiltonian K -manifold with proper moment map, so that $X//K$ is a locally free quotient and so an orbifold.

Definition 3.28. (Nodal infinite area vortices) An n -marked *nodal infinite area vortex* consists of the following datum:

- (a) (Stable map to the quotient) a stable r -marked pseudoholomorphic map $u : C_0 \rightarrow C \times X//K$ of class $([C], d_0)$ for some $d_0 \in H_2(X//K)$;
- (b) (Affine vortex bubbles) for each marking $z_j \in C_0$ a stable i_j -marked affine vortex $(C_j, A_j, u_j, \underline{z}_j)$ with markings $\underline{z}_j = (z_{j,1}, \dots, z_{j,i_j})$ and orbifold structure of C_j at the point at infinity $z_{j,0}$ matching the orbifold structure of C_0 at $z_{0,j} \in C_0$, so that the union $C_0 \cup C_1 \cup \dots \cup C_r$ is a balanced orbifold curve;

- (c) (Matching condition) the value $u_i(z_{i,0}) = p_2(u_i(z_{i,0}))$ in $X//K$, where $p_2 : C \times X//K \rightarrow X//K$ is the projection on the second factor.

An *isomorphism* of nodal infinite area vortices is a combination of an automorphism of the underlying curves intertwining the scalings and a bundle isomorphism on the affine vortex bubbles, intertwining the markings, scalings, connections and maps.

Remark 3.29. We denote by C the nodal curve obtained by gluing together the curves C_0 and C_i at $(z_i, z_{0,i})$; the matching condition means that the orbifold structures glue together to a twisting of C . Removal of singularities for vortices in Remark 3.22 implies that orbi-bundles P_j defined by the limit of the connections at infinity glue together with the bundle $P_0 \rightarrow C_0$ given by pull-back of $\Phi^{-1}(0) \rightarrow X//K$, to give a bundle $P \rightarrow C$. The matching condition then implies that the section u_0 over C_0 and u_j over C_j glue together to a section of $P(X)$ over C with fairly weak regularity properties at the nodes with infinite scaling, so that u takes values in the zero level set $\Phi^{-1}(0)$ on the subset of C with infinite scaling. An isomorphism of nodal symplectic vortices is then an automorphism of the domain intertwining the connections (singular at the nodes with infinite scaling), maps, scalings, and markings.

Let $\overline{M}_n^K(C, X)_\infty$ denote the moduli space of isomorphism classes of nodal infinite-area vortices. From the description above, $\overline{M}_n^K(C, X)_\infty$ is the union of fiber products

$$\overline{M}_r^K(C, X)_\infty \times_{(I_{X/K})^r} \prod_{j=1}^r \overline{M}_{|I_j|,1}^K(\mathbb{A}, X)$$

over unordered partitions $[I_1, \dots, I_r]$ of $\{1, \dots, n\}$.

We combine the moduli spaces above into a moduli space of vortices with varying area form. Let $\omega_C \in \Omega^2(C, \mathbb{R})$ be an area form.

Definition 3.30. (a) (Two-form corresponding to a scaling) Any scaled curve $(v : \hat{C} \rightarrow C, \lambda : \hat{C} \rightarrow \mathbb{P}(T_v^\vee \oplus \mathbb{C}))$ in the sense of Definition 2.44 defines a volume form $\omega_C(v, \lambda) \in \Omega^2(\hat{C}, \mathbb{R} \cup \{\infty\})$ as follows.

- (i) If λ is finite on the principal component, then $\omega_C(v, \lambda)$ is $|\lambda|^2$ times the form ω_C on the principal component, and on the bubble component $\omega_C(v, \lambda)$ vanishes.
 - (ii) If λ is infinite on the principal component, then $\omega_C(v, \lambda)$ is equal to $\lambda \wedge \overline{\lambda}$ on each bubble tree.
- (b) (Scaled vortices) A *marked scaled vortex* with domain C and target X consists of a marked scaled curve $v : \hat{C} \rightarrow C, \lambda : \hat{C} \rightarrow \mathbb{P}(T_v^\vee \oplus \mathbb{C})$ together with a vortex on \hat{C} corresponding to the area form $\omega_C(v, \lambda)$. That is,
- (i) If λ is finite on the principal component, then a vortex v_0 on the principal component corresponding to the area form $\omega_C(v, \lambda)|_{C_0}$ and a collection of sphere bubbles $v_i : C_i \rightarrow X$ for each component $C_i \subset \hat{C}, i \neq 0$;
 - (ii) If λ is infinite on the principal component, then a holomorphic map $u_0 : C_0 \rightarrow X//K$ on the principal component C_0 and a collection of nodal affine vortices on the bubble trees attached to C_0 ;

Both should satisfy natural matching conditions at the nodes. A marked scaled vortex is *polystable* if each non-principal component with finite, non-zero scaling (resp. zero or infinite scaling) has at least 2 (resp. 3) special points. The notion of *isomorphism* of

marked scaled vortices, using gauge transformations and automorphisms of the domain, is left to the reader. A marked scaled vortex is *stable* if it is polystable and has no infinitesimal automorphisms.

We denote by $\overline{M}_{n,1}^K(C, X)$ the moduli space of isomorphism classes of marked scaled vortices, and by $\overline{M}_{n,1}^K(C, X, d)$ the component with homology class $d \in H_2^K(X, \mathbb{Z})$. The notion of convergence of vortices extends naturally to the case of varying scaling, and defines a topology on $\overline{M}_{n,1}^K(C, X)$.

Theorem 3.31. (Properness of the moduli space of scaled symplectic vortices) *Let X be a Hamiltonian K -manifold with proper moment map convex at infinity equipped with a compatible almost complex structure, such that $X//K$ is a locally free quotient. Then $\overline{M}_{n,1}^K(C, X)$ is Hausdorff and the energy map $E : \overline{M}_{n,1}^K(C, X) \rightarrow [0, \infty)$ is proper.*

Proof. First we show that any sequence of vortices with varying scaling with bounded energy has a convergent subsequence, after passing to a limit. For sequences with scaling bounded away from infinity, this follows from Ott [77], who fixes a scaling but in the proof only uses upper bounds. For scalings approaching infinity, one obtains convergence in the absence of bubbling by the arguments of Gaio-Salamon [29]. As they show, bubbles in $X//K$ or X or affine vortices satisfy energy quantization and so bubbling happens only at finitely many points. Thus one obtains, on the complement of a finite set in C , an infinite area vortex and by Proposition 3.13 a stable pseudoholomorphic map to $C \times X//K$. By removal of singularities for orbifold stable maps, one obtains a stable map to $C \times X//K$. The bubbles connect argument in Ott [77] implies that the bubbles in X connect with the affine vortices, while the bubbles connect argument in Ziltener [98] shows that the affine vortex bubbles connect with the components mapping to $X//K$. Indeed Ziltener's theorem is stated for sequences of vortices on \mathbb{A} , but any sequence of vortices obtained by rescaling vortices on C with respect to an area form $|\rho|\omega_C$ with $|\rho| \rightarrow \infty$ satisfy the same estimates and so a similar proof holds. Let the closed sets be those for which any convergent sequence has a limit. That this defines a topology, and that this topology is Hausdorff, uses the construction of local distance functions as in (19) which will be left to the reader. \square

Remark 3.32. To prove the adiabatic limit Theorem 1.5, we will require that (A, u) is a vortex with respect to the area form $(\rho_0 + |\rho|)\omega_C$ for $\rho_0 \gg 0$. This is because without this adjustment the moduli space $\overline{M}_{n,1}^K(C, X)$ is almost never (even virtually) smooth, because of reducible pairs (A, u) , and so cannot be expected to yield invariants relating gauged Gromov-Witten invariants with Gromov-Witten invariants of the quotient. For a suitable “master space” relating the invariants for different ρ , see [33].

4. STACKS OF CURVES AND MAPS

A more sophisticated approach to moduli problems uses the language of *stacks*, in which spaces are replaced by categories and maps by functors. In this and the following section we show the spaces in the previous section are moduli spaces of stacks. This section contains mostly algebraic preliminaries, mainly a discussion of hom-stacks of morphisms of stacks, for which there is unfortunately no general theory yet. However, by a result of Lieblich [56, 2.3.4], the

stack of morphisms to a quotient stack by a reductive group is an Artin stack, and this is enough for our purposes. However, we also must show that the substacks of curves satisfying Mundet's semistability condition are algebraic, and this requires recalling some results of Schmitt [84] who gave a geometric invariant theory construction of a related moduli space.

By our convention a *stack* means a stack over the fppf site of schemes, following de Jong et al [20] which we take as our standard reference. (Another standard reference on stacks is Laumon-Moret-Bailly [52], with a correction by Olsson [71].) By abuse of terminology we say that a stack is a *scheme resp. algebraic space* if it is the stack associated to a scheme resp. algebraic space. A morphism of stacks $f : \mathcal{X} \rightarrow \mathcal{Y}$ is *representable* if for any morphism $g : S \rightarrow \mathcal{Y}$ where S is a scheme, the fiber product $S \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space. An *Artin stack* resp. *Deligne-Mumford stack* \mathcal{X} over a scheme S is a stack for which the diagonal $\mathcal{X} \rightarrow \mathcal{X} \times_S \mathcal{X}$ is representable, quasi-compact, and separated, and such that there exists an algebraic space X/S and a smooth resp. étale surjective morphism $X \rightarrow \mathcal{X}$. In characteristic zero (the only case considered here) an Artin stack is a Deligne-Mumford stack iff all the automorphism groups are finite, see e.g. [22, Remark 2.1]. A *gerbe* is a locally non-empty, locally connected (between any two objects exists a morphism) Artin stack. The category of Artin resp. Deligne-Mumford stacks is closed under disjoint unions and (category-theoretic) fiber products [52, 4.5]. A morphism of stacks is *proper* if it is separated, finite type, and universally closed.

4.1. Quotient stacks. The following are examples of stacks:

- Example 4.1.* (a) (Quot schemes) For integers r, n with $0 < r < n$ let $\mathrm{Gr}(r, n)$ denote the Grassmannian of subspaces of \mathbb{C}^n of dimension r . Any morphism from X to $\mathrm{Gr}(r, n)$ gives rise to a vector bundle $E \rightarrow X$ obtained by pull-back of the quotient bundle and a surjective morphism ϕ from the trivial bundle $X \times \mathbb{C}^n$ to E . Grothendieck [36], and later Olsson-Starr [72], studied such pairs in a very general setting, as part of a general program to construct moduli schemes for various functors. Given a scheme \mathcal{X}/S resp. separated Deligne-Mumford stack \mathcal{X}/S and a quasicoherent \mathcal{O}_X -module F , let $\mathrm{Quot}(F/\mathcal{X}/S)$ be the category that assigns to any S -scheme T the set of pairs (E, ϕ) where $\phi : F \times_S T \rightarrow E$ is a flat family of quotients. By Grothendieck [36] resp. Olsson-Starr [72], if \mathcal{X} is a scheme projective over S resp. Deligne-Mumford stack then $\mathrm{Quot}(F/\mathcal{X}/S)$ is a smooth scheme resp. algebraic space, whose connected components are projective resp. quasiprojective if \mathcal{X} is.
- (b) (Stack of coherent sheaves) The category of coherent sheaves on a projective scheme carries the structure of an Artin stack. If X is an S -scheme we denote by $\mathrm{Coh}(X/S)$ resp. $\mathrm{Vect}(X/S)$ the category that assigns to any $T \rightarrow S$ the category of coherent sheaves resp. vector bundles on $X \times_S T$. By [52, p. 29], see also [56, Theorem 2.1.1], if X is a projective scheme then $\mathrm{Coh}(X)$ is an Artin stacks and $\mathrm{Vect}(X)$ an open substack. Charts can be constructed as follows: after suitable twisting any sheaf $E \rightarrow X$ can be generated by its global sections, in which case E can be written as a quotient $F \rightarrow E$ where $F = X \times H^0(E)^\vee$. Then $\mathrm{Coh}(X)$ is isomorphic to $\mathrm{Quot}(F/X/S)$ locally near E to the quotient of $\mathrm{Quot}(F/X/S)$ by $\mathrm{Aut}(F)$.
- (c) (Stack of bundles, first version) Let G be a reductive group and X an S -scheme. We denote by $\mathrm{Bun}_G(X)$ the category that assigns to any $T \rightarrow S$ the category of principal G -bundles on $X \times_S T$. Then for any integer $r > 0$ the stack $\mathrm{Bun}_{GL(r)}(X)$ is canonically

isomorphic to the substack $\text{Vect}_r(X)$ of vector bundles of rank r . Any principal G -bundle corresponds to a $GL(V)$ -bundle $E \rightarrow X \times_S T$ together with a reduction of structure group $X \times_S T \rightarrow E/G$. If X is projective, then $\text{Vect}(X)$ is an Artin stack, being an open substack of the stack $\text{Coh}(X)$, and the above description gives that $\text{Bun}_G(X)$ is an Artin stack as in Sorger [85, 3.6.6 Corollary].

- (d) (Quotient stacks) If G is a reductive group scheme over S then we denote by BG the stack which assigns to an algebraic space $T \rightarrow S$ the category of principal G -bundles (torsors) over T . More generally if X is a G -scheme over S then X/G denotes the *quotient stack* which assigns to any morphism $T \rightarrow S$ the category of principal G -bundles $P \rightarrow T$ together with sections $u : T \rightarrow (P \times_S X)/G$, or equivalently equivariant morphisms from P to X , see e.g. de Jong et al [20, 55.12], Laumon-Moret-Bailly [52, 4.6.1]. In particular, $BG = \text{pt}/G$.

4.2. Stacks of curves.

Example 4.2. (a) (Stable curves) By a *nodal curve* over the scheme S we mean a flat proper morphism $\pi : C \rightarrow S$ of schemes such that the geometric fibers of π are reduced, one-dimensional and have at most ordinary double points (nodes) as singularities; that is, a nodal curve without the connectedness assumption in [10, Definition 2.1]. A nodal curve is *stable* if each fiber has no infinitesimal automorphisms. The category of connected nodal resp. stable marked curves of genus g is then a (non-finite-type) Artin resp. proper Deligne-Mumford stack $\overline{\mathfrak{M}}_{g,n}$ resp. $\overline{\mathcal{M}}_{g,n}$ [21], [10]. Let $\mathfrak{M}_{g,n,\Gamma}$ denote the stack consisting of objects whose combinatorial type in Definition 4.2 (2.1) is Γ . If Γ is connected then $\mathfrak{M}_{g,n,\Gamma}$ is a locally closed, local complete intersection Artin substack of $\overline{\mathfrak{M}}_{g,n}$ [10].

There is a canonical morphism $\overline{\mathfrak{M}}_{g,n,\Gamma} \rightarrow \overline{\mathcal{M}}_{g,n,\Gamma}$ which collapses unstable components, and can be defined as follows. Let $\pi : C \rightarrow S$ be an n -marked family of nodal curves. Let $\omega_{C/S}$ denote the relative dualizing sheaf over S and $\omega_{C/S}[z_1 + \dots + z_n]$ its twisting by $z_1 + \dots + z_n$. Consider the curve

$$C^{st} = \text{Proj } \oplus_{n \geq 0} \pi_*((\omega_{C/S}[z_1 + \dots + z_n])^{\otimes n}).$$

As noted in [10, Section 3], in the case that a family of n -marked curves arises from forgetting a marking of a family of $n+1$ -marked curves, each fiber C_s^{st} is obtained from C_s by collapsing unstable components. Furthermore, the formation of C^{st} commutes with base change and the map $C^{st} \rightarrow S$ is projective and flat. The general case is reduced to this one by adding markings locally. Let $\Upsilon : \Gamma \rightarrow \Gamma'$ be a morphism of modular graphs. Then there are morphisms of Artin resp. Deligne-Mumford stacks $\overline{\mathfrak{M}}(\Upsilon) : \overline{\mathfrak{M}}_{g,n,\Gamma} \rightarrow \overline{\mathfrak{M}}_{g,n,\Gamma'}$ resp. $\overline{\mathcal{M}}(\Upsilon) : \overline{\mathcal{M}}_{g,n,\Gamma} \rightarrow \overline{\mathcal{M}}_{g,n,\Gamma'}$. In the case of forgetting a tail, the morphism $\overline{\mathcal{M}}(\Upsilon)$ can be defined by the composition of the inclusion $\overline{\mathcal{M}}_{g,n,\Gamma} \rightarrow \overline{\mathfrak{M}}_{g,n,\Gamma}$, the map $\overline{\mathfrak{M}}(\Upsilon) : \overline{\mathfrak{M}}_{g,n,\Gamma} \rightarrow \overline{\mathfrak{M}}_{g,n,\Gamma'}$ and the collapsing map $\overline{\mathfrak{M}}_{g,n,\Gamma'} \rightarrow \overline{\mathcal{M}}_{g,n,\Gamma'}$. We denote by $\overline{\mathfrak{c}}_{g,n,\Gamma} \rightarrow \overline{\mathfrak{M}}_{g,n,\Gamma}$ resp. $\overline{\mathcal{C}}_{g,n,\Gamma} \rightarrow \overline{\mathcal{M}}_{g,n,\Gamma}$ the *universal curve* over $\mathfrak{M}_{g,n,\Gamma}$ resp. $\overline{\mathcal{M}}_{g,n,\Gamma}$ namely the category of n -marked nodal (resp. stable) curves equipped with an additional $n+1$ -marking which need not be distinct from the first n . In the case of $\overline{\mathcal{M}}_{g,n}$, the forgetful morphism f_{n+1} lifts to a map $\overline{\mathcal{C}}_{g,n+1} \rightarrow \overline{\mathcal{C}}_{g,n}$ and the section provided by the $n+1$ -st marking $\overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{C}}_{g,n+1}$ combine to an isomorphism $\overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{C}}_{g,n}$. In other words, $\overline{\mathcal{M}}_{g,n+1}$ can be considered the universal curve for $\overline{\mathcal{M}}_{g,n}$.

- (b) (Stable parametrized curves) Recall from section 2.2 that if C is a smooth connected projective curve then a C -parametrized curve is a map $u : \hat{C} \rightarrow C$ of homology class $[C]$ from a nodal curve \hat{C} to C , and is stable if it has only finitely many automorphisms. The category of nodal resp. stable C -parametrized curves forms an Artin stack $\overline{\mathfrak{M}}_n(C)$ resp. $\overline{\mathcal{M}}_n(C)$. More generally, for any rooted tree Γ we have Artin resp. Deligne-Mumford stacks $\overline{\mathfrak{M}}_{n,\Gamma}(C)$ resp. $\overline{\mathcal{M}}_{n,\Gamma}(C)$; the latter is a special case of the *Fulton-Macpherson compactification* studied in [28]. There is a morphism $\overline{\mathfrak{M}}_{n,\Gamma}(C) \rightarrow \overline{\mathcal{M}}_{n,\Gamma}(C)$ which collapses the unstable components. Indeed let $\pi : \hat{C} \rightarrow S, u : \hat{C} \rightarrow C$ be a family of C -parametrized n -marked nodal curves. Let L_C be an ample line bundle on C , and $\omega_{C/S}$ denote the relative dualizing sheaf over S and $\omega_{C/S}[z_1 + \dots + z_n]$ its twisting by $z_1 + \dots + z_n$. Consider the curve

$$(20) \quad C^{st} = \text{Proj } \oplus_{n \geq 0} \pi_*((\omega_{\hat{C}/S}[z_1 + \dots + z_n] \otimes u^* L_C^{\otimes 3})^{\otimes n}).$$

For families arising by forgetting markings, C^{st} is obtained from C by collapsing unstable components and the formation of C^{st} commutes with base change. The general case is reduced to this one by adding markings locally [10, Section 3].

Any morphism of rooted trees $\Upsilon : \Gamma \rightarrow \Gamma'$ of the type collapsing an edge, cutting an edge, forgetting a tail induces a morphism $\overline{\mathfrak{M}}(\Upsilon) : \overline{\mathfrak{M}}_{n,\Gamma}(C) \rightarrow \overline{\mathfrak{M}}_{n,\Gamma'}(C)$ resp. $\overline{\mathcal{M}}(\Upsilon) : \overline{\mathcal{M}}_{n,\Gamma}(C) \rightarrow \overline{\mathcal{M}}_{n,\Gamma'}(C)$. In the case of forgetting a tail, the morphism $\overline{\mathcal{M}}(\Upsilon)$ can be defined by the composition of the inclusion $\overline{\mathcal{M}}_{n,\Gamma}(C) \rightarrow \overline{\mathfrak{M}}_{n,\Gamma}(C)$, the map $\overline{\mathfrak{M}}(\Upsilon) : \overline{\mathfrak{M}}_{n,\Gamma}(C) \rightarrow \overline{\mathfrak{M}}_{n,\Gamma'}(C)$ followed by the collapsing map $\overline{\mathfrak{M}}_{n,\Gamma'}(C) \rightarrow \overline{\mathcal{M}}_{n,\Gamma'}(C)$.

- (c) (Curves with scalings) Let S be an algebraic space over \mathbb{C} and C a smooth projective nodal curve over S . Recall from e.g. [5, p.95] that the dualizing sheaf $\omega_{C/S}$ is locally free. Factor the projection $\pi : C \rightarrow S$ locally into the composition of a regular embedding $i : C \rightarrow R$ of relative dimension m and a smooth morphism $j : R \rightarrow S$ of relative dimension l . The normal sheaf $N_{C/R}$ of i is locally free of rank m , while the sheaf of relative Kähler differentials $\Omega_{R/S}^1$ is locally free of l . The relative dualizing sheaf of π is

$$\omega_{C/S} := (\Lambda^m N_{C/R})^{-1} \otimes \Lambda^l \Omega_{R/S}^1.$$

Explicitly, if \tilde{C} denotes the normalization of C (the disjoint union of the irreducible components of C) with nodal points $\{\{w_1^+, w_1^-\}, \dots, \{w_k^+, w_k^-\}\}$ then $\omega_{\tilde{C}}$ is the sheaf of sections of $\omega_{\tilde{C}} := T^\vee \tilde{C}$ whose residues at the points w_j^+, w_j^- sum to zero, for $j = 1, \dots, k$. Denote by $\mathbb{P}(\omega_{C/S} \oplus \mathbb{C})$ the fiber bundle obtained by adding in a section at infinity. A *scaling* of C is a section λ of $\mathbb{P}(\omega_{C/S} \oplus \mathbb{C})$. The category of pairs (C, λ) is an Artin stack, with charts given by the forgetful morphisms from stable curves with additional marked points, equipped with scalings.

- (d) (Stable scaled affine lines) Let S be an algebraic space over \mathbb{C} . A *nodal n -marked scaled affine line*, see Section 2.20, consists of a smooth projective nodal curve C over S , an $n + 1$ -tuple (z_0, \dots, z_n) of sections $S \rightarrow C$ (the markings) distinct from the nodes and each other, and a scaling λ of $\mathbb{P}(\omega_{C/S} \oplus \mathbb{C})$, satisfying the following conditions:
- (i) (Affine structure on each component on which it is non-degenerate) on each irreducible component C_i of C the form λ is either zero, infinite, or finite except for a single order two pole at a node of C .

- (ii) (Monotonicity) on each non-self-crossing path of components from $z_i, i > 0$ to z_0 , there is exactly one component on which λ is finite and non-zero; on the components before resp. after, λ vanishes resp. is infinite.

The first condition means that on the complement of the pole, if it exists, there is a canonical affine structure. An n -marked scaled curve is *stable* if it has no infinitesimal automorphisms, or equivalently, each component with degenerate resp. non-degenerate scaling has at least three resp. two special points. We denote by $\overline{\mathcal{M}}_{n,1}(\mathbb{A})$ resp. $\overline{\mathfrak{M}}_{n,1}(\mathbb{A})$ the stack of stable resp. nodal affine scaled n -marked curves; this is a proper complex variety resp. Artin stack. The former was constructed in [61]. Charts for the latter are given by forgetful morphisms $\overline{\mathcal{M}}_{m,1}(\mathbb{A}) \rightarrow \overline{\mathfrak{M}}_{n,1}(\mathbb{A})$ for $m > n$ given by forgetting the last $m - n$ points, as in the case without scaling in Behrend-Manin [10].

We also wish to allow *twistings* at nodes of C with infinite scaling, see [75, Section 2] for a precise definition: the node has a cyclic automorphism group μ_r and there exist charts for neighborhoods of the node in each component of the form U/μ_r acting by inverse roots of unity. As in [75, Theorem 1.8], the category $\overline{\mathfrak{M}}_{n,1}^{\text{tw}}(\mathbb{A})$ of scaled twisted marked curves is equivalent to the category of scaled log twisted marked curves, compatibly with base change, and this implies that $\overline{\mathfrak{M}}_{n,1}^{\text{tw}}(\mathbb{A})$ is an Artin stack. For any colored tree Γ we denote by $\overline{\mathfrak{M}}_{n,1,\Gamma}^{\text{tw}}(\mathbb{A})$ resp. $\overline{\mathcal{M}}_{n,1,\Gamma}^{\text{tw}}(\mathbb{A})$ the stack of nodal resp. stable scaled n -marked affine lines of combinatorial type Γ .

There is a canonical morphism $\overline{\mathfrak{M}}_{n,1}(\mathbb{A}) \rightarrow \overline{\mathcal{M}}_{n,1}(\mathbb{A})$ defined as follows. Let $(C, \lambda, \underline{z})$ be a family of scaled affine lines over an algebraic space S . Let Λ denote the sheaf over C that assigns to an open subset $U \subset C$ the space of (possibly infinite) sections of $T^\vee U$ given by $f\lambda$ where $f \in \mathcal{O}_C(U)$ is regular on U . Thus Λ is rank one on the components where $\lambda \notin \{0, \infty\}$, and is rank zero otherwise. The sum

$$\omega_{C/S}^\lambda[z_1 + \dots + z_n] = \omega_{C/S}[z_1 + \dots + z_n] + \Lambda$$

is then a subsheaf of the sheaf of the projectivized relative dualizing sheaf on C . In terms of the normalization \tilde{C}_s of any fiber C_s , $\omega_{C/S}^\lambda[z_1 + \dots + z_n]$ is the sheaf of relative differentials with poles at the markings, nodes, and an additional pole on any component with finite scaling at the node connecting with a component with infinite scaling. Consider the curve

$$(21) \quad C^{st} = \text{Proj} \oplus_{n \geq 0} \pi_*((\omega_{C/S}^\lambda[z_1 + \dots + z_n])^{\otimes n}).$$

In the case that C arises from a family obtained by forgetting a marked point on a stable scaled affine curve, C^{st} collapses unstable components and its formation commutes with base change. This construction collapses the bubbles that are unstable *furthest away from the root marking*, in particular, any colored component that becomes unstable after forgetting the marking. However, the adjacent component may be destabilized by collapses of this component; it is then necessary to apply the construction again to collapse this component. See Figure 11 which shows an example with three components of infinite scaling and four components with finite scaling, each with a single marking, after one of the markings is forgotten. The forgetful morphism is produced by applying the Proj construction *twice*, in contrast to the case of stable curves where a single application suffices. The general case is reduced to this one by adding markings locally.

Any morphism of colored trees $\Upsilon : \Gamma \rightarrow \Gamma'$ of the type *collapsing an edge, collapsing edges with relations, cutting an edge, cutting an edge with relations or forgetting a tail* induces morphisms $\overline{\mathfrak{M}}(\Upsilon) : \overline{\mathfrak{M}}_{n,1,\Gamma}(\mathbb{A}) \rightarrow \overline{\mathfrak{M}}_{n,1,\Gamma'}(\mathbb{A})$ and $\overline{\mathcal{M}}(\Upsilon) : \overline{\mathcal{M}}_{n,1,\Gamma}(\mathbb{A}) \rightarrow \overline{\mathcal{M}}_{n,1,\Gamma'}(\mathbb{A})$. In the case of forgetting a tail, the morphism $\overline{\mathcal{M}}(\Upsilon)$ can be defined by the composition of the inclusion $\overline{\mathcal{M}}_{n,1,\Gamma}(\mathbb{A}) \rightarrow \overline{\mathfrak{M}}_{n,1,\Gamma}(\mathbb{A})$, the map $\overline{\mathfrak{M}}(\Upsilon) : \overline{\mathfrak{M}}_{n,1,\Gamma}(\mathbb{A}) \rightarrow \overline{\mathfrak{M}}_{n,1,\Gamma'}(\mathbb{A})$ followed by the collapsing map $\overline{\mathfrak{M}}_{n,1,\Gamma'}(\mathbb{A}) \rightarrow \overline{\mathcal{M}}_{n,1,\Gamma'}(\mathbb{A})$.

By its construction, the stack $\overline{\mathcal{M}}_{n,1}(\mathbb{A})$ has a *universal curve* $\overline{\mathcal{C}}_{n,1}(\mathbb{A}) \rightarrow \overline{\mathcal{M}}_{n,1}(\mathbb{A})$ equipped with universal scaling and markings. The forgetful morphism $\overline{\mathcal{M}}_{n+1,1}(\mathbb{A}) \rightarrow \overline{\mathcal{M}}_{n,1}(\mathbb{A})$ is isomorphic to the universal curve, as in the Knudsen case, given by the map $\overline{\mathcal{M}}_{n+1,1}(\mathbb{A}) \rightarrow \overline{\mathcal{C}}_{n,1}(\mathbb{A})$ given by the n -marked curve (21) with section given by the image of the $n+1$ -st marked point. The inverse $\overline{\mathcal{C}}_{n,1}(\mathbb{A}) \rightarrow \overline{\mathcal{M}}_{n+1,1}(\mathbb{A})$ is given on the level of geometric points by a consideration of various cases: If the extra marked point is a smooth point on a component with infinite scaling, distinct from the other markings, then one adds a bubble with finite scaling with the additional marking to the curve. If the extra marked point is a smooth point on a component with finite or zero scaling, distinct from the other marking, then one adds that point as an additional marking. If the extra marked point coincides with one of the other markings, or with a node, the one adds an additional bubble component with the appropriate scaling, and puts the additional marking on that component. This shows that the morphism $\overline{\mathcal{M}}_{n+1,1}(\mathbb{A}) \rightarrow \overline{\mathcal{C}}_n(\mathbb{A})$ induces a bijection of geometric points and is therefore (as a morphism of nodal curves over $\overline{\mathcal{M}}_{n,1}(\mathbb{A})$) an isomorphism.

More generally, one may consider stacks $\overline{\mathfrak{M}}_{n,s}(\mathbb{A})$ of s -scaled n -marked affine lines, that is, curves equipped with markings z_1, \dots, z_n and scalings $\lambda_1, \dots, \lambda_s$. By similar arguments, these stacks are Artin and the stacks of stable curves $\overline{\mathcal{M}}_{n,s}(\mathbb{A})$ are Deligne-Mumford.

- (e) (Stacks of scaled curves) A *family of nodal C -parametrized curves with finite scaling* consists of $\pi : \hat{C} \rightarrow S$ a family of nodal curves, $u : \hat{C} \rightarrow C$ a family of nodal maps of homology class $[C]$, a family of sections $z_1, \dots, z_n : S \rightarrow \hat{C}$, and a section $\lambda : \hat{C} \rightarrow \mathbb{P}(\omega_{\hat{C}/C} \oplus \mathbb{C})$ of the projectivized relative dualizing sheaf (see 4.2 (c)) satisfying the following conditions:

- (i) (Finite on any marking) $\lambda(z_i)$ is finite.
- (ii) (Scaling on each bubble component) on each component C_i of \hat{C} mapping to a point in C , $\lambda|_{C_i}$ has a unique pole of order 2, at the node connecting C_i with the principal component C_0 .
- (iii) (Monotonicity) on each non-self-crossing path of components from the principal component to the component containing $z_i, i > 0$, there is exactly one component on which λ is finite and non-zero; on the components before resp. after, λ vanishes resp. is infinite.

The category of such forms an Artin stack $\mathfrak{M}_{\Gamma,n,1}(C)$. There is a “forgetful morphism” from $\mathfrak{M}_{n,1}(C)$ to $\mathfrak{M}_n(C)$ which forgets the scaling. There is also a morphism $\mathfrak{M}_{n,1}(C) \rightarrow \overline{\mathcal{M}}_{n,1}(C)$ collapsing the unstable components, whose construction is a combination of the previous cases and left to the reader. Similarly, $\overline{\mathfrak{M}}_{n,1}^{\text{tw}}(C)$ is the stack of scaled

marked curves with log structures at the nodes with infinite scaling as in [75, Section 2].

4.3. Stacks of morphisms. Many of our examples will arise as stacks of morphisms between stacks. Fix an algebraic space S . Let \mathcal{X} and \mathcal{Y} be Artin stacks over S . Let $\mathrm{Hom}_S(\mathcal{X}, \mathcal{Y})$ be the fibered category over the category of S -schemes, which to any $T \rightarrow S$ associates the groupoid of functors $\mathcal{X} \times_S T \rightarrow \mathcal{Y} \times_S T$. Unfortunately, there seems to be no general construction which guarantees that $\mathrm{Hom}_S(\mathcal{X}, \mathcal{Y})$ is an Artin stack, but partial results are given by Olsson [74], Aoki [4], Romagny [82], and Lieblich [56, 2.3.4].

Example 4.3. (a) (Hom stacks between schemes) If X, Y are projective schemes over a Noetherian scheme S with X flat over S then $\mathrm{Hom}_S(X, Y)$ is representable by a quasiprojective S -scheme (a subscheme of the Hilbert scheme) by Grothendieck's construction of Hilbert schemes, described in [24].

- (b) (Stable maps) Let $\overline{\mathfrak{M}}_{g,n}$ denote the stack of nodal curves with genus g and n markings from Example 4.2, $\overline{\mathcal{C}}_{g,n} \rightarrow \overline{\mathfrak{M}}_{g,n}$ the universal curve, and X a projective variety. Then $\overline{\mathfrak{M}}_{g,n}(X) := \mathrm{Hom}_{\overline{\mathfrak{M}}_{g,n}}(\overline{\mathcal{C}}_{g,n}, X)$ is the stack of *nodal (or prestable) maps to X* . The locus $\overline{\mathcal{M}}_{g,n}(X)$ of stable maps is defined as the sub-stack of maps with no infinitesimal automorphisms, or equivalently, such that each component on which the map is constant of genus zero (resp. one) has at least three (resp. one) special point. By the constructions in Behrend-Manin [10] and Fulton-Pandharipande [27], $\overline{\mathfrak{M}}_{g,n}(X)$ resp. $\overline{\mathcal{M}}_{g,n}(X)$ is an Artin resp. proper Deligne-Mumford stack. Similarly for any type Γ , let $\overline{\mathfrak{M}}_{g,n,\Gamma}(X) = \mathrm{Hom}_{\overline{\mathfrak{M}}_{g,n,\Gamma}}(\overline{\mathcal{C}}_{g,n,\Gamma}, X)$ denote the compactified stack of maps of combinatorial type Γ and $\overline{\mathcal{M}}_{g,n,\Gamma}(X)$ the locus of stable maps. Then $\overline{\mathfrak{M}}_{g,n,\Gamma}(X)$ resp. $\overline{\mathcal{M}}_{g,n,\Gamma}(X)$ is an Artin resp. proper Deligne-Mumford stack. There is a canonical morphism from $\overline{\mathfrak{M}}_{g,n,\Gamma}(X)$ to $\overline{\mathcal{M}}_{g,n,\Gamma}(X)$ which collapses unstable components. Indeed, given a family $u : C \rightarrow X$ and an ample line bundle $L \rightarrow X$ consider the curve

$$(22) \quad C^{st} = \mathrm{Proj} \bigoplus_{n \geq 0} \pi_*(\omega_{C/S}(z_1 + \dots + z_n) \otimes u^* L^3)^{\otimes n}.$$

For families arising from forgetting markings from a family of stable maps, C^{st} is obtained from C by collapsing unstable markings, and the formation of C commutes with base change. The general case reduces to this one by adding markings locally [10].

Any morphism $\Upsilon : \Gamma \rightarrow \Gamma'$ of type cutting an edge, collapsing an edge, or forgetting a tail induces morphisms of moduli stacks

$$\overline{\mathfrak{M}}(\Upsilon, X) : \overline{\mathfrak{M}}_{g,n,\Gamma}(X) \rightarrow \overline{\mathfrak{M}}_{g,n,\Gamma'}(X), \quad \overline{\mathcal{M}}(\Upsilon, X) : \overline{\mathcal{M}}_{g,n,\Gamma}(X) \rightarrow \overline{\mathcal{M}}_{g,n,\Gamma'}(X).$$

In the first case the morphism is induced from fiber product with the morphism $\overline{\mathfrak{M}}(\Upsilon, X) : \overline{\mathfrak{M}}_{g,n,\Gamma} \rightarrow \overline{\mathfrak{M}}_{g,n,\Gamma'}$, while $\overline{\mathcal{M}}_{g,n,\Upsilon}$ is defined by composing the inclusion $\overline{\mathcal{M}}(\Upsilon, X) \rightarrow \overline{\mathfrak{M}}(\Upsilon, X)$ with $\overline{\mathfrak{M}}(\Upsilon, X)$ and the collapsing morphism to $\overline{\mathcal{M}}_{g,n,\Gamma'}(X)$.

- (c) (Stacks of bundles, second version) Let X be an S -scheme and G a reductive group. Let $\mathrm{Hom}(X, BG)$ be the category that assigns to $T \rightarrow S$ the groupoid of G -bundles on $X \times_S T$. Then $\mathrm{Hom}(X, BG)$ is a stack, naturally isomorphic to the stack $\mathrm{Bun}_G(X)$ of G -bundles. In particular, if X is a projective S -scheme then $\mathrm{Hom}(X, BG)$ is an Artin stack by Example 4.1.

- (d) (Stacks of morphisms to quotient stacks) Let X be an algebraic space over S , G a reductive group and Y a G -scheme, and Y/G the quotient stack. Let $\mathrm{Hom}_S(X, Y/G)$ denote the category that assigns to any $T \rightarrow S$, a principal G -bundle P over $X \times_S T$ and a section $X \times_S T \rightarrow P \times_G Y$. By Lieblich [56, 2.3.4], $\mathrm{Hom}_S(X, Y/G)$ is an Artin stack. More generally if $f : \mathcal{X} \rightarrow \mathcal{Z}$ is a proper morphism of Artin stacks and Y is a separated and finitely-presented G -scheme then let $\mathrm{Hom}_{\mathcal{Z}}(\mathcal{X}, Y/G)$ be the fibered category that associates to any morphism $T \rightarrow \mathcal{Z}$ and object X of $\mathcal{X} \times_{\mathcal{Z}} T$ the category of pairs (P, u) where $P \rightarrow X$ is a principal G -bundle and section $u : X \rightarrow P \times_G Y$. By Olsson [2, Lemma C.5], $\mathrm{Hom}_{\mathcal{Z}}(\mathcal{X}, Y/G)$ is an Artin stack.
- (e) (Stacks of morphisms to quotient stacks as quotients) In this example following Schmitt [84] we describe a realization of morphisms to quotient stacks as substacks of Grothendieck's quot stack discussed in Example 4.1. First suppose that C is a scheme over S , $G = GL(n)$ and $X = \mathbb{P}^{n-1}$. Any morphism $u : C \rightarrow X/G$ corresponds to a vector bundle $E \rightarrow C$ together with a section of the projectivization $\mathbb{P}(E)$: There are several equivalent descriptions of this data: (i) a vector bundle E and a line sub-bundle $L := u^* \mathcal{O}_{\mathbb{P}(E)}(1) \rightarrow C$, (ii) a vector bundle E^\vee , a line bundle L^\vee , and a surjective morphism $E^\vee \rightarrow L^\vee$. The latter datum is termed a *swamp* (short for *sheaf with map*) or more generally, a *bump* (short for *bundle with map*) if the group G is arbitrary reductive. Schmitt [84] shows that the functor from schemes to sets which associates to any scheme the set of isomorphism classes of stable bumps, can be realized as a git quotient of a quot scheme. The *type* of a bump is the pair of integers $(\deg(E), \deg(L))$.
- If S is an arbitrary scheme, then a bump over C parametrized by S with representation V consists of a principal G -bundle P on $S \times C$, a line bundle $L \rightarrow S \times C$, and a homomorphism φ from $P(V)$ to L . Since $\mathrm{Bun}_{\mathbb{C}^\times}(C)$ splits non-canonically as $\mathrm{Jac}(C) \times BC^\times$, this data is equivalent to a morphism $S \rightarrow \mathrm{Jac}(C)$ together with a line bundle on the parameter space S , which is the formulation adopted in Schmitt [84].
- The idea of Schmitt's construction of the moduli space of semistable bumps is as follows. After suitable twisting, we may assume that E is generated by its global sections, in which case E^\vee is a quotient of a trivial vector bundle F and we obtain a double quotient $F \rightarrow E^\vee \rightarrow L^\vee$. This gives a quotient $F^2 \rightarrow E^\vee \times L^\vee$ with the property that the map to L^\vee factors through E^\vee , and so a point in the quot scheme $\mathrm{Quot}(F/X/S)^2$. Let $\mathfrak{M}^{G, \mathrm{quot}, \mathrm{fr}}(C, X, F)$ denote the open substack of $\mathrm{Quot}(F/X/S)^2$ arising in this way, and let $\mathfrak{M}^{G, \mathrm{quot}}(C, X, F) = \mathfrak{M}^{G, \mathrm{quot}, \mathrm{fr}} / \mathrm{Aut}(F)$ be the quotient stack by the action of the general linear group $\mathrm{Aut}(F)$. Let $\overline{\mathfrak{M}}^{G, \mathrm{quot}}(C, X, F)$ be its closure in $\mathrm{Quot}(F/X/S)^2 / \mathrm{Aut}(F)$. More generally, for any reductive group G and projective G -variety Y , a choice of representation $G \rightarrow GL(V)$ and embedding $Y \rightarrow \mathbb{P}(V)$ gives a stack $\overline{\mathfrak{M}}^{\mathrm{quot}, G}(C, X; F)$ by taking the closure of $\mathrm{Hom}(X, Y/G)$ (or rather, those maps whose bundles are quotients of F) in $\mathrm{Quot}(F/X/S)^2$.
- (f) (Inertia stacks) The *inertia stack* of a Deligne-Mumford (or Artin) stack \mathcal{X} is

$$I_{\mathcal{X}} := \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X}$$

where both maps are the diagonal. The objects of $I_{\mathcal{X}}$ may be identified with pairs (x, g) where $x \in \mathcal{X}$ and $g \in \mathrm{Aut}_{\mathcal{X}}(x)$. For example, if $\mathcal{X} = X/G$ is a global quotient by a

finite group then

$$I_{\mathcal{X}} = \cup_{[g] \in G/\text{Ad}(G)} X^g / Z_g$$

where $G/\text{Ad}(G)$ denotes the set of conjugacy classes in X and Z_g is the centralizer of g . There is also an interpretation as a hom stack (see e.g. [1])

$$I_{\mathcal{X}} = \cup_{r>0} I_{\mathcal{X},r}, \quad I_{\mathcal{X},r} := \text{Hom}^{\text{rep}}(B\mu_r, \mathcal{X}).$$

- (g) (Rigidified inertia stacks) The following stack plays an important role in Gromov-Witten theory of Deligne-Mumford stacks as developed by Abramovich-Graber-Vistoli [1]. If μ_r is the group of r -th roots of unity then $B\mu_r$ is an Deligne-Mumford stack. If \mathcal{X} is a Deligne-Mumford stack then

$$\bar{I}_{\mathcal{X}} = \cup_{r>0} \bar{I}_{\mathcal{X},r}, \quad \bar{I}_{\mathcal{X},r} := I_{\mathcal{X}/r} / B\mu_r.$$

is the *rigidified inertia stack* of representable morphisms from $B\mu_r$ to \mathcal{X} , see [1]. There is a canonical quotient cover $\pi : I_{\mathcal{X}} \rightarrow \bar{I}_{\mathcal{X}}$ which is r -fold over $\bar{I}_{\mathcal{X},r}$ which acts on cohomology isomorphism

$$\pi^* H^*(\bar{I}_{\mathcal{X}}, \mathbb{Q}) \rightarrow H^*(I_{\mathcal{X}}, \mathbb{Q})$$

so for the purposes of defining orbifold Gromov-Witten invariants, $\bar{I}_{\mathcal{X}}$ can be replaced by $I_{\mathcal{X}}$ at the cost of additional factors of r on the r -twisted sectors. If $\mathcal{X} = X/G$ is a global quotient of a scheme X by a finite group G then

$$\bar{I}_{X/G} = \coprod_{(g)} X^{\text{ss},g} / (Z_g / \langle g \rangle)$$

where $\langle g \rangle \subset Z_g$ is the cyclic subgroup generated by g .

- (h) (Rigidified inertia stacks for locally free git quotients) Suppose that X is a polarized smooth projective G -variety such that $X//G$ is locally free. Then

$$I_{X//G} = \coprod_{(g)} X^{\text{ss},g} / Z_g$$

where $X^{\text{ss},g}$ is the fixed point set of $g \in G$ on X^{ss} , Z_g is its centralizer, and the union is over all conjugacy classes,

$$\bar{I}_{X//G} = \coprod_{(g)} X^{\text{ss},g} / (Z_g / \langle g \rangle)$$

where $\langle g \rangle$ is the (finite) group generated by g .

- (i) (Stacks of nodal gauged maps) Consider the Artin stack $\overline{\mathfrak{M}}_{g,n}$ of marked nodal curves and X/G the quotient stack associated to the quotient of a projective scheme X by a reductive group G . Then $\text{Hom}_{\overline{\mathfrak{M}}_{g,n}}(\overline{\mathfrak{C}}_{g,n}, X/G)$ is an Artin stack.
- (j) (Stacks of parameterized nodal gauged maps) Let C be a curve and X a G -scheme. Then $\text{Hom}_{\overline{\mathfrak{M}}_n(C)}(\overline{\mathfrak{C}}_n(C), X/G)$ is the category that assigns to a morphism $T \rightarrow S$ the groupoid of marked nodal curves $\hat{C} \rightarrow T \times C$, of class $[C]$ on the second factor, equipped with a principal G -bundle $P \rightarrow \hat{C}$ and a morphism $C \rightarrow P \times_G X$.

- (k) (Stacks of parameterized nodal affine gauged maps) Let X be a G -scheme. Then $\mathrm{Hom}_{\mathfrak{M}_{n,1}(\mathbb{A})}(\overline{\mathfrak{C}}_{n,1}(\mathbb{A}), X/G)$ is the category that assigns to a morphism $T \rightarrow S$ the groupoid of marked affine nodal curves $\hat{C} \rightarrow T$ equipped with a principal G -bundle $P \rightarrow \hat{C}$ and a morphism $C \rightarrow P \times_G X$. More generally, $\mathrm{Hom}_{\overline{\mathfrak{M}}_{n,1}^{\mathrm{tw}}(\mathbb{A})}(\overline{\mathfrak{C}}_{n,1}^{\mathrm{tw}}(\mathbb{A}), X/G)$ is the hom-stack allowing orbifold singularities in the domain at the nodes with infinite scaling.

4.4. Twisted stable maps. We recall the definitions of twisted curve and twisted stable map to a Deligne-Mumford stack from Abramovich-Graber-Vistoli [1] and Abramovich-Olsson-Vistoli [2]. These definitions are needed for the construction of the moduli stack of affine gauged maps in the case that $X//G$ is an orbifold, but not if the quotient is free. Denote by μ_r the group of r -th roots of unity.

Definition 4.4. (Twisted curves) Let S be a scheme. An n -marked twisted curve over S is a collection of data $(f : \mathcal{C} \rightarrow S, \{\ddagger_i \subset \mathcal{C}\}_{i=1}^n)$ such that

- (a) (Coarse moduli space) \mathcal{C} is a proper stack over S whose geometric fibers are connected of dimension 1, and such that the coarse moduli space of \mathcal{C} is a nodal curve over S .
- (b) (Markings) The $\ddagger_i \subset \mathcal{C}$ are closed substacks that are gerbes over S , and whose images in \mathcal{C} are contained in the smooth locus of the morphism $\mathcal{C} \rightarrow S$.
- (c) (Automorphisms only at markings and nodes) If $\mathcal{C}^{ns} \subset \mathcal{C}$ denotes the *non-special locus* given as the complement of the \ddagger_i and the singular locus of $\mathcal{C} \rightarrow S$, then $\mathcal{C}^{ns} \rightarrow \mathcal{C}$ is an open immersion.
- (d) (Local form at smooth points) If $p \rightarrow \mathcal{C}$ is a geometric point mapping to a smooth point of \mathcal{C} , then there exists an integer r , equal to 1 unless p is in the image of some \ddagger_i , an étale neighborhood $\mathrm{Spec}(R) \rightarrow \mathcal{C}$ of p and an étale morphism $\mathrm{Spec}(R) \rightarrow \mathrm{Spec}_S(\mathcal{O}_S[x])$ such that the pullback $\mathcal{C} \times_S \mathrm{Spec}(R)$ is equal to $\mathrm{Spec}(R[z]/z^r = x)/\mu_r$.
- (e) (Local form at nodal points) If $p \rightarrow \mathcal{C}$ is a geometric point mapping to a node of \mathcal{C} , then there exists an integer r , an étale neighborhood $\mathrm{Spec}(R) \rightarrow \mathcal{C}$ of p and an étale morphism $\mathrm{Spec}(R) \rightarrow \mathrm{Spec}_S(\mathcal{O}_S[x, y]/(xy - t))$ for some $t \in \mathcal{O}_S$ such that the pullback $\mathcal{C} \times_S \mathrm{Spec}(R)$ is equal to $\mathrm{Spec}(R[z, w]/zw = t', z^r = x, w^r = y)/\mu_r$ for some $t' \in \mathcal{O}_S$.

Let \mathcal{X} be a smooth Deligne-Mumford stack proper over a scheme S over a field of characteristic zero with projective coarse moduli space X , or an open subset thereof.

Definition 4.5. A *twisted stable map* from an n -marked twisted curve $(\pi : \mathcal{C} \rightarrow S, (\ddagger_i \subset \mathcal{C})_{i=1}^n)$ over S to \mathcal{X} is a representable morphism of S -stacks $u : \mathcal{C} \rightarrow \mathcal{X}$ such that the induced morphism on coarse moduli spaces $u_c : C \rightarrow X$ is a stable map in the sense of Kontsevich [49] from the n -pointed curve $(C, \underline{z} = (z_1, \dots, z_n))$ to X , where z_i is the image of \ddagger_i . The *homology class* of a twisted stable curve is the homology class $u_*[\mathcal{C}_s] \in H_2(X, \mathbb{Q})$ of any fiber \mathcal{C}_s .

Twisted stable maps naturally form a 2-category, but every 2-morphism is unique and invertible if it exists, and so this 2-category is naturally equivalent to a 1-category which forms a stack over schemes [1].

Theorem 4.6. ([1, 4.2]) *The stack $\overline{\mathcal{M}}_{g,n}(\mathcal{X})$ of twisted stable maps from n -pointed genus g curves into \mathcal{X} is a Deligne-Mumford stack. If \mathcal{X} is proper, then for any $c > 0$ the union of substacks $\overline{\mathcal{M}}_{g,n}(\mathcal{X}, d)$ with homology class $d \in H_2(\mathcal{X}, \mathbb{Q})$ satisfying $(d, [\omega]) < c$ is proper.*

The proof uses the equivalence of the category of twisted curves with log-twisted curves. Let $\overline{I}_{\mathcal{X}}$ denote the rigidified inertia stack as in Proposition 4.3 (g). The moduli stack of twisted stable maps admits evaluation maps

$$\mathrm{ev} : \overline{\mathcal{M}}_{g,n}(\mathcal{X}) \rightarrow \overline{I}_{\mathcal{X}}, \quad \overline{\mathrm{ev}} : \overline{\mathcal{M}}_{g,n}(\mathcal{X}) \rightarrow \overline{I}_{\mathcal{X}}^n,$$

where the second is obtained by composing with the involution $\overline{I}_{\mathcal{X}} \rightarrow \overline{I}_{\mathcal{X}}$ induced by the map $\mu_r \rightarrow \mu_r, \zeta \mapsto \zeta^{-1}$. There is a modification of the definition which produces evaluation maps to the unrigidified moduli stacks: let $\overline{\mathcal{M}}_{g,n}^{\mathrm{fr}}(\mathcal{X})$ denote the moduli space of *framed* twisted stable maps, that is, twisted stable maps with sections of the gerbes at the marked points [1]. These stacks are $\prod_{i=1}^n r_i$ -fold covers of $\overline{\mathcal{M}}_{g,n}(\mathcal{X})$, where $r_i : \overline{\mathcal{M}}_{g,n}(\mathcal{X}) \rightarrow \mathbb{Z}_{\geq 0}$ is the order of the isotropy group at the i -th marking, and admit evaluation maps

$$\mathrm{ev}^{\mathrm{fr}} : \overline{\mathcal{M}}_{g,n}^{\mathrm{fr}}(\mathcal{X}) \rightarrow I_{\mathcal{X}}^n, \quad \overline{\mathrm{ev}}^{\mathrm{fr}} : \overline{\mathcal{M}}_{g,n}^{\mathrm{fr}}(\mathcal{X}) \rightarrow I_{\mathcal{X}}^n.$$

If \mathcal{C} is a finite disjoint union of twisted curves, then a stable map from \mathcal{C} to \mathcal{X} is a stable map of each of its components. For any possibly disconnected combinatorial type Γ , we denote by $\overline{\mathcal{M}}_{g,n,\Gamma}(\mathcal{X})$ resp. $\overline{\mathcal{M}}_{g,n,\Gamma}^{\mathrm{fr}}(\mathcal{X})$ the stack stable maps resp. framed stable maps whose underlying stable map of schemes has combinatorial type Γ .

Proposition 4.7. (a) (Cutting an edge) *If Γ' is obtained from Γ by cutting an edge, there is a morphism*

$$(23) \quad \mathcal{G}(\Upsilon, X) : \overline{\mathcal{M}}_{g,n,\Gamma'}(\mathcal{X}) \times_{\overline{I}_{\mathcal{X}}^2} \overline{I}_{\mathcal{X}} \rightarrow \overline{\mathcal{M}}_{g,n,\Gamma}(\mathcal{X})$$

where the second morphism is the diagonal $\Delta : I_{\mathcal{X}} \rightarrow I_{\mathcal{X}}^2$, and an isomorphism

$$\overline{\mathcal{M}}_{g,n,\Gamma'}^{\mathrm{fr}}(\mathcal{X}) \times_{I_{\mathcal{X}}^2} I_{\mathcal{X}} \rightarrow \overline{\mathcal{M}}_{g,n,\Gamma}^{\mathrm{fr}}(\mathcal{X}).$$

(b) (Collapsing an edge) *If Γ' is obtained from Γ by collapsing an edge then there is an isomorphism*

$$\overline{\mathcal{M}}_{g,n,\Gamma'}(\mathcal{X}) \times_{\overline{\mathfrak{M}}_{n,\Gamma'}} \overline{\mathfrak{M}}_{g,n,\Gamma} \rightarrow \overline{\mathcal{M}}_{g,n,\Gamma}(\mathcal{X})$$

and similarly for framed twisted stable maps.

Example 4.8. (Inertia stacks for toric orbifolds) Consider a stack $\mathcal{X} = X//G$ obtained as the quotient of a vector space X by a torus G with weights μ_1, \dots, μ_k at a weight $\nu \in \mathfrak{g}^{\vee}$, see (29). For each subset $\{\mu_i, i \in I\}$ with $\nu \in \mathrm{span}\{\mu_i, i \in I\}$, let Λ_I denote the lattice generated by the $\mu_i, i \in I$, and $G_I = \exp(\Lambda_I^{\vee})$ the subgroup generated by the dual lattice. Let X_I denote the span of the weight spaces for $\mu_i, i \in I$ and $X_I//G$ the git quotient of X_I by G . For any element $g \in G$ let $I(g)$ denote the set of i such that $g \in \ker(\exp(\mu_i) : G \rightarrow \mathbb{C}^{\times})$. Then

$$I_{X//G} = \cup_{g \in G} (X_{I(g)}//G)$$

(finite by the previous paragraph) and is therefore a union of toric varieties. For each $i \in I(g)$, vanishing of the coordinate in $X_{I(g)}$ corresponding to i defines a divisor \tilde{D}_i , whose (possibly empty) image in $X_{I(g)}$ is a divisor $D_{i,g}$. The cohomology of $I_{X//G}$ is generated by the divisors $D_{i,g}, i \in I(g)$.

4.5. Coarse moduli spaces. A *coarse moduli space* for a stack \mathcal{X} is an algebraic space X together with a morphism $\pi : \mathcal{X} \rightarrow X$ such that π induces a bijection between the geometric points of X and \mathcal{X} and π is universal for maps to algebraic spaces. By a theorem of Keel-Mori [45], coarse moduli spaces for Artin stacks with finite inertia (in particular, Deligne-Mumford stacks in characteristic zero) exist.

Proposition 4.9. *A Deligne-Mumford stack over \mathbb{C} is proper iff its coarse moduli space is proper iff its coarse moduli space is compact and Hausdorff in the analytic topology.*

Proof. The first equivalence is e.g. [74, 2.10]. The second equivalence is folklore, see for example [76, Theorem 3.17]. \square

Artin [6] has given conditions for a category fibered in groupoids to be an Artin stack. In particular, each object should admit a versal deformation, universal if the stack is Deligne-Mumford. Versal deformations give a notion of *topological convergence* of a sequence of objects in the category, defined if the corresponding sequence of points s_ν in the parameter space S for a versal deformation converges to a point s , in which case the limit is the equivalence class of objects corresponding to s . In particular, these notions define the underlying C^0 topology of the coarse moduli spaces.

Example 4.10. (a) (Convergence of nodal curves) Any projective nodal curve has a versal deformation given in the analytic category by a simple gluing construction via *Schiffer variations* in which small balls around the nodes are removed and the components glued together via maps $z \mapsto \delta/z$ [5, p. 176] in local coordinates z near the node.

(b) (Convergence of stable maps) Any map from a projective nodal curve to a projective variety has a versal deformation given by considering its graph as an element in a suitable Hilbert scheme of subvarieties, see for example [24]. The construction of the Hilbert scheme is reduced to the construction of a Quot scheme, which in turn reduces to representability of the Grassmannian. For the Grassmannian topological convergence of a subbundle implies topological convergence in the sense described above for (uni)versal deformations. It follows that topological convergence for maps is the usual notion of convergence of stable maps discussed in, for example, McDuff-Salamon [62].

(c) (Convergence of bundles) Any vector bundle over a curve has a versal deformation given by considering it, after twisting by a sufficiently positive line bundle, as a quotient of a trivial bundle, and the same holds for principal bundles for reductive groups by considering them as vector bundles with reductions. Topological convergence of a sequence of isomorphism classes of bundles is the usual notion of topological convergence of holomorphic bundles, that is, C^0 convergence of the corresponding holomorphic structures on the components after complex gauge transformation.

(d) (Convergence of isomorphism classes of vector bundles) In particular, let C be a smooth projective curve and $\mathcal{M} = \text{Bun}^{\text{ss}}(C, d, r)$ the moduli stack of semistable bundles of homology class $d \in H_2^G(C, \mathbb{Z})$ and rank $r \geq 0$. By a theorem of Narasimhan-Seshadri [69], if stable=semistable then the coarse moduli space M for \mathcal{M} admits a homeomorphism ϕ to its image in the moduli space of unitary representations of the fundamental group $R = \text{Hom}(\pi_1(C), U(r))/U(r)$, where here $\text{Hom}(\pi_1(C), U(r))$ denotes the topological space of representations of $\pi_1(C)$ in $U(r)$. The Hilbert scheme construction, or the construction of universal families in [69] shows that inverse map $R \rightarrow M$ is continuous.

5. STABLE GAUGED MAPS

In this section we identify the moduli space of symplectic vortices constructed in Section 3 as the coarse moduli space of a substack of the moduli stack of gauged maps satisfying a semistability condition introduced by Mundet [66] and further studied by Schmitt [83], [84]. This correspondence of Hitchin-Kobayashi type implies that the moduli space of symplectic vortices, if every vortex has finite automorphism group, is the moduli space of a proper Deligne-Mumford stack.

5.1. Gauged maps. Let G be a complex reductive group, X be a smooth projective G -variety and C a smooth connected projective curve. In this section we construct the stack $\overline{\mathfrak{M}}_n^G(C, X)$ of n -marked gauged maps for integers $n \geq 0$.

Definition 5.1. An n -marked nodal gauged map from C to X over a scheme S is a morphism $u : \hat{C} \rightarrow C \times X/G$ from a nodal curve \hat{C} over S whose projection onto the first factor has homology class $[C]$, such that if $C_{i,s} \subset \hat{C}_s$ is a component that maps to a point in C , then the bundle corresponding to $u|_{C_i}$ is trivial. More explicitly, such a morphism is given by a datum $(\hat{C}, P, u, \underline{z})$ where

- (a) (Nodal curve) $\hat{C} \rightarrow S$ is a proper flat morphism with reduced nodal curves as fibers;
- (b) (Bundle over the principal component) $P \rightarrow C \times S$ is a principal G -bundle;
- (c) (Section of the associated fiber bundle) $u : \hat{C} \rightarrow P(X) := (P \times X)/G$ is a family of stable maps with base class $[C]$, that is, the composition of u with the projection $P(X) \rightarrow C$ has class $[C]$.

A *morphism* between gauged maps (S, \hat{C}, P, u) and (S', \hat{C}', P', u') consists of a morphism $\beta : S \rightarrow S'$, a morphism $\phi : P \rightarrow (\beta \times 1)^* P'$, and a morphism $\psi : \hat{C} \rightarrow \hat{C}'$ such that the first diagram below is Cartesian and the second and third commute:

$$\begin{array}{ccccc}
 \hat{C} & \longrightarrow & S & & P & \longrightarrow & S \times C & & \hat{C} & \xrightarrow{u} & P(X) \\
 \psi \downarrow & & \downarrow \beta & & \phi \downarrow & & \downarrow \text{id} & & \downarrow \psi & & \downarrow [\phi \times \text{id}_X] \\
 \hat{C}' & \longrightarrow & S' & & (\beta \times 1)^* P' & \longrightarrow & S \times C & & \hat{C}' & \xrightarrow{u'} & P'(X).
 \end{array}$$

An n -marked nodal gauged map is equipped with an n -tuple $(z_1, \dots, z_n) \in \hat{C}^n$ of distinct smooth points on \hat{C} .

Let $\overline{\mathfrak{M}}_n^G(C, X)$ denote the category of n -marked nodal gauged maps, $\overline{\mathfrak{M}}_n^{G, st}(C, X)$ the subcategory where $u : \hat{C} \rightarrow P(X)$ is a stable map, and $\mathfrak{M}_n^G(C, X)$ the subcategory where $\hat{C} \rightarrow C$ is an isomorphism, that is, the domain is irreducible. The functor from $\overline{\mathfrak{M}}_n^G(C, X)$ to schemes which assigns to any datum $(S, C, P, u, \underline{z})$ the base scheme S makes $\overline{\mathfrak{M}}_n^G(C, X)$ resp. $\overline{\mathfrak{M}}_n^{G, st}(C, X)$ resp. $\mathfrak{M}_n^G(C, X)$ into a category fibered in groupoids. We denote by $\overline{\mathfrak{C}}_n^G(C, X) \rightarrow \overline{\mathfrak{M}}_n^G(C, X)$ the universal curve, consisting of a datum $(P \rightarrow C, u : \hat{C} \rightarrow P(X), z : S \rightarrow \hat{C})$ with z not necessarily mapping to the smooth locus of \hat{C} . The universal curve maps canonically to X/G

via evaluation at z' :

$$\bar{\mathfrak{C}}_n^G(C, X) \rightarrow X/G, \quad (S, C, P, \underline{z}, z') \mapsto (S, (\pi \circ u \circ z')^* P, u \circ z').$$

Theorem 5.2. $\bar{\mathfrak{M}}_n^G(C, X)$ resp. $\bar{\mathfrak{M}}_n^{G, st}(C, X)$ resp. $\mathfrak{M}_n^G(C, X)$ is a (non-finite-type, non-separated) Artin stack.

Proof. If $\bar{\mathfrak{C}}_n(C) \rightarrow \bar{\mathfrak{M}}_n(C)$ is the universal curve, $\text{Hom}_{\bar{\mathfrak{M}}_n(C)}(\bar{\mathfrak{C}}_n(C), X/G)$ is an Artin stack by the results of Section 4.3 (d). The stack $\bar{\mathfrak{M}}_n^G(C, X)$ is the substack of $\text{Hom}_{\bar{\mathfrak{M}}_n(C)}(\bar{\mathfrak{C}}_n(C), X/G)$ corresponding to morphisms $f : \hat{C} \rightarrow X/G$ such that on each component \hat{C}_i mapping to a point in C , the principal G -bundle $P_i \rightarrow \hat{C}_i$ defined by f is trivial. Since triviality on the bubbles is an open condition, $\bar{\mathfrak{M}}_n^G(C, X)$ is an Artin stack as well. The condition that $u : \hat{C} \rightarrow P(X)$ is stable (has no infinitesimal automorphisms) is an open condition, hence $\bar{\mathfrak{M}}_n^{G, st}(C, X)$ is an open substack, hence also an Artin stack. Similarly the locus $\mathfrak{M}_n^G(C, X)$ where $\hat{C} \cong C$ is open and so also Artin. \square

Lemma 5.3. (Existence of a morphism collapsing unstable components) *There is a morphism $\bar{\mathfrak{M}}_n^G(C, X) \rightarrow \bar{\mathfrak{M}}_n^{G, st}(C, X)$ collapsing unstable components. The composition $\bar{\mathfrak{M}}_n^{G, st}(C, X) \rightarrow \bar{\mathfrak{M}}_{n-1}^G(C, X) \rightarrow \bar{\mathfrak{M}}_{n-1}^{G, st}(C, X)$ collapsing unstable components is isomorphic to the universal curve $\bar{\mathfrak{C}}_{n-1}^G(C, X) \rightarrow \bar{\mathfrak{M}}_{n-1}^G(C, X)$, and in particular proper.*

Proof. Given a nodal curve $\pi : \hat{C} \rightarrow S$ with dualizing sheaf $\omega_{\hat{C}/S}$, a morphism $u : \hat{C} \rightarrow C \times P(X)$, an ample G -line bundle $L \rightarrow X$, and an ample line bundle $L_C \rightarrow C$. The formation of the curve

$$\hat{C}^{st} = \text{Proj} \bigoplus_{n \geq 0} \pi_*(\omega_{\hat{C}/S}(z_1 + \dots + z_n) \otimes u^*(L_C \boxtimes P(L))^{\otimes 3})^{\otimes n}$$

commutes with base change, in the case that the family arises from a stable family by forgetting a marking. Then u factors through \hat{C}^{st} and this gives the required family in this case. The general case reduces to this one by adding markings locally, see Behrend-Manin [10, Theorem 3.10] and [10, Proposition 4.6]. \square

5.2. Mundet stability. Gauged maps corresponding to solutions of the vortex equations correspond to maps satisfying a semistability condition introduced by Mundet [66]. In this section we construct the stack $\bar{\mathcal{M}}_n^G(C, X)$ of Mundet-semistable gauged maps. These are used later to define gauged Gromov-Witten invariants.

First recall some terminology from the study of moduli spaces of G -bundles, from Ramanathan [81]. We restrict here to the case that G is connected. A subgroup $R \subset G$ is *parabolic* if G/R is complete. Given a parabolic subgroup, the maximal reductive *Levi subgroup* $L \subset G$ is unique up to isomorphism and R admits a decomposition $R = LU$ where U is a maximal unipotent subgroup. The quotient map will be denoted $p : R \rightarrow R/U \cong L$ and the inclusion $i : L \rightarrow G$. A *parabolic reduction* of a bundle P to R is a section $\sigma : C \rightarrow P/R$.

Definition 5.4. (Associated Graded Bundle) Let P be a principal G -bundle on a curve C .

- (a) (As an induced bundle) Given a parabolic reduction $\sigma : C \rightarrow P/R$, let σ^*P denote the associated R bundle, $p_*\sigma^*P$ the associated L -bundle, and $j : R \rightarrow G$ the inclusion. The bundle $\text{Gr}(P) := j_*p_*\sigma^*P$ is the *associated graded G -bundle* for σ .
- (b) (As a degeneration) Let $\sigma : C \rightarrow P/R$ be parabolic reduction, $Z(L)$ denote the center of the Levi subgroup L , $\mathfrak{z}(\mathfrak{l})$ its Lie algebra, and $\lambda \in \mathfrak{z}(\mathfrak{l})$ a generic antidominant coweight (with respect to the roots of the Lie algebra \mathfrak{p} of P restricted to $\mathfrak{z}(\mathfrak{l})$). For $z \in \mathbb{C}^\times$, the induced family of automorphisms ϕ of R by $z^\lambda = \exp(\ln(z)\lambda)$ by conjugation induces a family of bundles $P_{\sigma,\lambda} := j_*((\sigma^*P \times \mathbb{C}) \times_\phi R)$ over $C \times \mathbb{C}$ with central fiber $\text{Gr}(P)$.

Example 5.5. (Associated graded bundles for vector bundles) If $G = GL(n)$, then a parabolic reduction is equivalent to a filtration of the associated vector bundle and $\text{Gr}(P)$ is the frame bundle of the associated graded vector bundle, see [81]. The degeneration in the second definition above is the one that deforms the “off diagonal” parts of the transition maps of P to zero.

Definition 5.6. (Associated Graded Section) Given a section $u : C \rightarrow P(X)$, define the *associated graded section* $\text{Gr}(u) : \hat{C} \rightarrow (\text{Gr}(P))(X)$ associated to (σ, λ) as the unique stable limit u_0 of the sections u_z of $P_{\sigma,\lambda}|_{C \times z}(X)$ given by acting on u by z^λ .

The Mundet stability condition is a collection of inequalities given by integrals over the curve C , analogous to the definition of stability of vector bundles by degrees of sub-bundles. Suppose λ is a weight of $Z(L)$ and so defines a one-dimensional representation \mathbb{C}_λ . Via the trivialization $\mathfrak{z}(\mathfrak{l}) \cong (p_*\sigma^*P)(\mathfrak{z}(\mathfrak{l})) \subset (\text{Gr}(P))(\mathfrak{g})$ the element λ defines an infinitesimal automorphism of $\text{Gr}(P)$, fixing the principal component $\text{Gr}(u)_0$ of $\text{Gr}(u)$. The polarization $\mathcal{O}_X(1)$ defines a line bundle $P(\mathcal{O}_X(1)) \rightarrow P(X)$ and the infinitesimal automorphism defined by λ acts on the fibers over $\text{Gr}(u)_0$ with a weight $\mu_X(\text{Gr}(u)_0, \lambda)$.

Definition 5.7. (Mundet weight) The *Mundet weight* of the pair (σ, λ) as above is defined by

$$(24) \quad \mu(\sigma, \lambda) = \int_{[C]} c_1(p_*\sigma^*P \times_L \mathbb{C}_{-\lambda}) + \mu_X(\text{Gr}(u)_0, -\lambda)[\omega_C].$$

A gauged map (P, u) is *Mundet stable* iff it satisfies the

$$(25) \quad (\text{Weight Condition}) \quad \mu(\sigma, \lambda) < 0$$

for all (σ, λ) , *Mundet unstable* if there exists a *de-stabilizing pair* (σ, λ) violating (25) with strict inequality, *Mundet semistable* if it is not unstable, and *Mundet polystable* if it is semistable but not stable and (P, u) is isomorphic to its associated graded for any pair (σ, λ) satisfying the above with equality. A gauged map is *semistable* if it is Mundet semistable with stable section, and *stable* if it is semistable and has finite automorphism group.

Remark 5.8. (a) (Connection with stability of bundles) In the case that X is trivial, Mundet stability is the same as Ramanan stability of principal G -bundles [81].

- (b) (Definition in terms of the moment map) If $P(K)$ is a smooth principal K -bundle so that $P = P(K) \times_K G$ is a smooth principal G -bundle, then via the correspondence between complex structures on $P(G)$ and connections on P we may view $\text{Gr}(P)$ as a limiting connection on $P(K)$, and the section $\text{Gr}(u)$ as a stable section of $P(K) \times_K X$. Then the weight $\mu_X(\text{Gr}(u)_0, \lambda)$ can be expressed in terms of the moment map as

$$\mu_X(\text{Gr}(u)_0, \lambda) = ((P(K))(\Phi) \circ \text{Gr}(u)_0, \lambda),$$

by the usual correspondence between moment maps and linearizations of actions.

- (c) (Dependence on choices) The stability condition depends on the cohomology class $[\omega_C] \in H^2(C)$, in addition to the metric on \mathfrak{k} and the choice of moment map (or polarization) on X . Rescaling the metric on \mathfrak{k} is equivalent to rescaling $[\omega_C]$ or to rescaling the moment map. Allowing a varying curve C equipped with a cohomology class $[\omega_C]$ leads to various properness issues, see Mundet-Tian [68].
- (d) (Comparison with Mundet's definition) We have chosen the definition to generalize that of Ramanathan [81] for principal G -bundles. Mundet's definition in [66, Section 4] is slightly different: For a parabolic reduction σ and possibly irrational antidominant $\lambda \in \mathfrak{z}$, identified with an infinitesimal gauge transformation,

$$\mu(\sigma, \lambda) = \inf_t \int_C (F_{e^{it\lambda}A}, -\lambda) + ((e^{it\lambda}u)^*P(\Phi), -\lambda)\omega_C.$$

Then $\mu(\sigma, \lambda)$ agrees with the previous definition in the case that λ is a coweight, since in this case the infimum equals the limit as $t \rightarrow -\infty$, the right-hand-side of (24). To see that the two definitions are the same, it suffices to check that if (25) is violated by some irrational λ it is also violated for rational λ . For λ' sufficiently close to λ and defining the same parabolic reduction, we have

$$(26) \quad \lim_{t \rightarrow \infty} e^{it\lambda} A = e^{it\lambda'} A =: A_\infty, \quad \lim_{t \rightarrow \infty} F_{e^{it\lambda}A} = \lim_{t \rightarrow \infty} F_{e^{it\lambda'}A} = F_{A_\infty}$$

uniformly in all derivatives. Furthermore, by Gromov compactness $e^{it\xi}u$ Gromov converges to some limit $u_\infty : \hat{C} \rightarrow P(X)$ as $t \rightarrow \infty$, with principal component $u_{0,\infty} : \hat{C} \rightarrow P(X)$ and

$$\lim_{t \rightarrow \infty} \int_C ((e^{it\lambda}u)^*P(\Phi), -\lambda)\omega_C \rightarrow \int_C (u_{0,\infty}^*P(\Phi), -\lambda)\omega_C.$$

In addition, λ stabilizes the limiting pair (A_∞, u_∞) . Let $\lambda' \in \mathfrak{k}(P)_{(A_\infty, u_\infty)}$ commute with λ . It follows from the local slice theorem for holomorphic actions that if λ' is sufficiently close to λ then for z not in the bubbling set

$$(27) \quad \lim_{t \rightarrow \infty} e^{it\lambda'} u(z) = u_{0,\infty}(z) = \lim_{t \rightarrow \infty} e^{it\lambda} u(z).$$

Indeed after passing to a maximal torus containing both λ, λ' the equation (27) holds iff the weights for the action at $u_{0,\infty}(z)$ have the same sign on λ and λ' . Since rational Lie algebra vectors are dense in the Lie algebra of any closed subgroup, we may find λ' rational satisfying (26), (27) and so violating semistability if λ does. Mundet also allows a correction coming from the center of \mathfrak{g} on the right-hand-side of (25), so that in the case X trivial and $G = GL(n)$ the definition agrees with that for vector bundles.

Theorem 5.9 below gives the equivalence of the stability condition with the existence of a solution to the vortex equations. We denote by $\mathcal{G}(P)$ the group of *complex gauge transformations* of P . There is a one-to-one correspondence between the space $\mathcal{J}(P(G))$ of complex structures on $P(G)$ and connections $\mathcal{A}(P)$ on P . The identification $\mathcal{A}(P) \rightarrow \mathcal{J}(P(G))$ is equivariant for the action of $\mathcal{K}(P)$, in the sense that $J_{kA} = Dk \circ J_A \circ Dk^{-1}$. Thus, the identification defines an extension of the $\mathcal{K}(P)$ action on $\mathcal{A}(P)$ to an action of $\mathcal{G}(P)$. The group $\mathcal{G}(P)$ acts on $\mathcal{H}(P, X)$ by composition on the second factor.

Theorem 5.9 (Mundet’s Hitchin-Kobayashi correspondence for vortices [66]). *Let $P \rightarrow C$ be a principal K -bundle. A pair $(A, u) \in \mathcal{H}(P, X)$ defines a Mundet-stable gauged map if and only if there exists a complex gauge transformation $g \in \mathcal{G}(P)$ such that $g(A, u)$ is a vortex.*

Remark 5.10. (Analytic Mundet stability) Mundet’s proof depends on the convexity of the functional $\mathcal{I}(P)$ (depending on the choice of (A, u)) obtained by integrating the one form determined by the moment map

$$(28) \quad \mathcal{I}(P) : \mathcal{G}(P)/\mathcal{K}(P) \rightarrow \mathbb{R}, \quad [\exp(it\xi)] \mapsto \int_0^1 \langle F_{\exp(it\xi)(A, u)}, \xi \rangle dt.$$

If (A, u) is complex gauge equivalent to a symplectic vortex, then $\mathcal{I}(P)$ is bounded from below. On the other hand, if $\mathcal{I}(P)$ is not bounded from below then Mundet (using previous results of Uhlenbeck-Yau [91]) constructs a direction $\xi \in \mathfrak{k}(P)$ in which $\lim_{t \rightarrow \infty} \exp(-it\xi)(A, u)$ exists and

$$\lim_{t \rightarrow \infty} (F_{\exp(-it\xi)(A, u)}, \xi) \geq 0$$

and shows that the corresponding parabolic reduction violates the stability condition. A pair (A, u) is Mundet unstable resp. semistable resp. stable iff the Mundet functional $\mathcal{I}(P)$ is not bounded from below resp. bounded from below resp. attains its minimum in $\mathcal{G}(P)/\mathcal{K}(P)$.

The algebraic moduli spaces arising from the Mundet semistability condition are investigated in Schmitt [84]. Let $\mathcal{M}_n^G(C, X) \subset \mathfrak{M}_n^{G, st}(C, X)$ denote the category of n -marked gauged maps to X with irreducible domain that are Mundet semistable. We wish to show that $\mathcal{M}_n^G(C, X)$ is an Artin stack, for which it suffices to show that the semistability condition is open. Recall from Section 4.3 (d) Schmitt’s compactification of the moduli space of gauged maps by *projective bumps*. Schmitt [84] defines a semistability condition for projective bumps, which generalizes Mundet semistability for gauged maps. Let $\overline{\mathfrak{M}}^{G, \text{quot}}(C, X, F)$ denote the moduli stack of projective bumps, let $\overline{\mathcal{M}}^{G, \text{quot}}(C, X, F)$ denote the subcategory of $\overline{\mathfrak{M}}^{G, \text{quot}}(C, X, F)$ consisting of families of Mundet semistable bumps. For $d \in H_2^G(X, \mathbb{Z})$ and $\overline{\mathcal{M}}^{G, \text{quot}}(C, X, F, d)$ the moduli substack of $\overline{\mathcal{M}}^{G, \text{quot}}(C, X, F)$ of semistable bumps with class d . The semistable locus $\overline{\mathcal{M}}^{G, \text{quot}}(C, X, F)$ is independent of F for F of sufficiently large rank, by [84, Section 2.7], and will be denoted $\overline{\mathcal{M}}^{G, \text{quot}}(C, X)$. Recall that X is equipped with the equivariant class $[\omega_{X, G}] \in H_G^2(X)$.

Theorem 5.11. *Let X be a smooth polarized projective G -variety. The moduli stack of projective bumps $\overline{\mathfrak{M}}^{G, \text{quot}}(C, X, F)$ resp. Mundet semistable projective bumps $\overline{\mathcal{M}}^{G, \text{quot}}(C, X, F)$ is an Artin stack locally of finite type, containing $\mathcal{M}^G(C, X)$ as an open substack. More precisely each $\overline{\mathfrak{M}}^{G, \text{quot}}(C, X, F, d)$ resp. $\overline{\mathcal{M}}^{G, \text{quot}}(C, X, F, d)$ has a presentation as a quotient of a closed subscheme of a quot scheme resp. semistable locus in a closed subscheme of a quot scheme. If stable=semistable for projective bumps then for each constant $c > 0$, the union of components $\overline{\mathcal{M}}^{G, \text{quot}}(C, X, F, d)$ with $(d, [\omega_{X, G}]) < c$ is a proper Deligne-Mumford stack with projective coarse moduli space.*

Proof. Schmitt [84] avoids the language of stacks, but the construction is the same: $\overline{\mathfrak{M}}^{G, \text{quot}}(C, X, F)$ is the quotient of a rigidified moduli space $\overline{\mathfrak{M}}^{G, \text{quot}, \text{fr}}(C, X, F)$, and $\overline{\mathcal{M}}^{G, \text{quot}}(C, X, F)$ is the

quotient of the git semistable locus [84, Theorem 2.7.1.4]). The necessary local quot scheme is constructed as follows, in the case $G = GL(n)$. Let $E \rightarrow C$ be a vector bundle and $u : E \rightarrow L$ a quotient corresponding to a section of $\mathbb{P}(E^\vee)$. After suitable twisting, we may assume that E is generated by its global sections, in which case E is a quotient of a trivial vector bundle F . We then obtain a double quotient $F \rightarrow E \rightarrow L$. Such a double quotient can be considered as a quotient $F^2 \rightarrow E \times L$. Let $\mathfrak{M}^{G, \text{fr}, \text{quot}}(C, X, F)$ denote the open subscheme of the quot scheme $\text{Quot}_{F^2/C}$ consisting of such quotients. Let $\mathfrak{M}^{G, \text{quot}}(C, X, F) = \mathfrak{M}^{G, \text{fr}, \text{quot}}(C, X, F) / \text{Aut}(F)$ be the quotient stack by the action of the general linear group $\text{Aut}(F)$. Let $\overline{\mathfrak{M}}^{G, \text{quot}}(C, X, F)$ its closure in $\text{Quot}_{F^2/C} / \text{Aut}(F)$. Schmitt [84, Section 2.7] shows that a suitable canonical polarization on $\text{Quot}_{F^2/C}$ gives a semistability condition which reproduces Mundet semistability. The git construction shows that each substack $\overline{\mathcal{M}}^{G, \text{quot}}(C, X, F, d)$ is proper. On the other hand, the set of classes d such that $(d, [\omega_{X, G}]) < c$ and $\overline{\mathcal{M}}^{G, \text{quot}}(C, X, F, d)$ is non-empty, is finite, since, as one may check, $(d, [\omega_{X, G}])$ is the degree of the line bundle L in Schmitt's construction. \square

On the other hand, there is a natural Kontsevich-style moduli space which allows bubbling in the fibers satisfying a stability condition. Denote by $\overline{\mathcal{M}}_n^{\text{pre}, G}(C, X)$ denote the category of n -marked nodal gauged maps (that is, not necessarily stable sections) that are Mundet semistable. Let $\overline{\mathcal{M}}_n^G(C, X)$ denote the subcategory of $\overline{\mathcal{M}}_n^{\text{pre}, G}(C, X)$ consisting of Mundet semistable gauged maps that are semistable, that is, have sections that are stable maps, and $\mathcal{M}_n^G(C, X)$ the subcategory where $\hat{C} \cong C$. The relationship between the Kontsevich-style compactification $\overline{\mathcal{M}}^G(C, X)$ and the Grothendieck-style compactification $\overline{\mathcal{M}}^{G, \text{quot}}(C, X)$ is given by relative version of Givental's collapsing morphism [31, p. 646]:

Proposition 5.12. *Let X be a smooth polarized projective G -variety. Then $\overline{\mathcal{M}}_n^G(C, X)$ is an Artin stack equipped with a proper Deligne-Mumford morphism to $\overline{\mathcal{M}}^{G, \text{quot}}(C, X)$.*

Proof. By Givental's main lemma [31, p. 646], see [55, Lemma 2.6], [25, Section 8], [12], [79] for any smooth projective variety X embedded in a projective space $\mathbb{P}(V)$, there exists a proper morphism $\overline{\mathcal{M}}_{g,0}(C \times X) \rightarrow \text{Quot}_{F/C}$ where $\text{Quot}_{F/C}$ is the quot scheme of the trivial bundle $F = C \times V^\vee$ compactifying $\text{Hom}(C, \mathbb{P}(V))$. We apply this as follows, continuing Example 4.2 (e): Consider the forgetful morphism $\overline{\mathcal{M}}_0^G(C, X) \rightarrow \text{Hom}(C, BG)$. As in Narasimhan-Seshadri [69] for $G = GL(n, \mathbb{C})$ or Ramanan [81], Sorger [85] in general, the stack $\text{Hom}(C, BG)$ admits a local presentation as a quotient $\mathfrak{M}^{\text{fr}, \text{quot}}(C, F) / \text{Aut}(F)$ where $\mathfrak{M}^{\text{fr}, \text{quot}}(C, F)$ is a quasiprojective scheme of bundles whose associated vector bundle is equipped with a presentation as a quotient of F . The space $\mathfrak{M}^{\text{fr}, \text{quot}}(C, F)$ has a universal G -bundle $\mathcal{U}^{\text{fr}, \text{quot}}(C, F) \rightarrow C \times \mathfrak{M}^{\text{fr}, \text{quot}}(C, F)$ equipped with a G -equivariant $\text{Aut}(F)$ -action. Let $\tilde{d} \in H_2(\mathcal{U}^{\text{quot}}(C, F) \times_G X)$ be the class corresponding to d that is, whose push-forward under $\mathcal{U}^{\text{fr}, \text{quot}}(C, F) \times_G X \rightarrow \mathfrak{M}^{\text{quot}}(C, F)$ is zero and whose fiber class is determined by d . The stack $\mathfrak{M}^G(C, X, F, d)$ is a category of bundles with section, and so is isomorphic to the quotient of the rigidified moduli space $\mathcal{M}_{g,0}(\mathcal{U}^{\text{fr}, \text{quot}}(C, F) \times_G X, \tilde{d})$ by the action of $\text{Aut}(F)$. Let $\overline{\mathfrak{M}}^{\text{fr}, \text{quot}}(C, \mathcal{U}^{\text{quot}}(C, F) \times_G X, \tilde{d})$ be the subscheme of $\text{Quot}_{F^2/C}$ compactifying morphisms $C \rightarrow \mathcal{U}^{\text{quot}}(C, F) \times_G X$ of class \tilde{d} . By the relative version of Givental's lemma [79, Theorem, p.4] there exists a proper morphism

$g : \overline{\mathcal{M}}_{g,0}(\overline{\mathfrak{U}}^{\text{quot}}(C, F) \times_G X, \tilde{d}) \rightarrow \overline{\mathfrak{M}}^{\text{fr,quot}}(C, \overline{\mathfrak{U}}^{\text{quot}}(C, F) \times_G X, \tilde{d})$ mapping each stable map to the corresponding quotient. The morphism $\overline{\mathfrak{M}}^G(C, X, F, d) \rightarrow \overline{\mathfrak{M}}^{G,\text{quot}}(C, X, F, d)$ is the quotient of g by the action of $\text{Aut}(F)$. Since g is proper and of Deligne-Mumford type, so is the quotient. After restricting to the semistable locus, we may assume that F is sufficiently large so that every bundle occurs as a quotient of F . Then $\overline{\mathcal{M}}_0^G(C, X)$ is the inverse image of the open substack $\overline{\mathcal{M}}^{G,\text{quot}}(C, X)$ and so also an Artin stack. Furthermore $\overline{\mathcal{M}}_n^G(C, X)$ is the inverse image of $\overline{\mathcal{M}}_0^G(C, X)$ under the forgetful morphism obtained by iterating Lemma 5.3, and so an Artin stack. Since the forgetful morphism and g are both Deligne-Mumford and proper, the claim follows. \square

Corollary 5.13. *Let X be a smooth polarized projective G -variety. Suppose that every Mundet semistable gauged map is stable. For each constant $c > 0$, the union of components $\overline{\mathcal{M}}_n^G(C, X, d)$ with $(d, [\omega_{X,G}]) < c$ is a proper Deligne-Mumford stack.*

Proof. By Theorem 5.11 and Proposition 5.12, the morphisms $\overline{\mathcal{M}}^{G,\text{quot}}(C, X, d) \rightarrow \text{pt}$ and $\overline{\mathcal{M}}_n^G(C, X, d) \rightarrow \overline{\mathcal{M}}^{G,\text{quot}}(C, X, d)$ are proper and Deligne-Mumford, hence so is their composition. \square

There is another approach to the properness result above which uses symplectic geometry rather than the git constructions in Schmitt [84]. Let K be a maximal compact subgroup of G .

Theorem 5.14. *Let X be a smooth polarized projective G -variety or a G -vector space with a proper moment map. Suppose that every semistable gauged map is stable. The map assigning to any stable gauged map the corresponding vortex defines a homeomorphism Z from the coarse moduli space of $\overline{\mathcal{M}}_n^G(C, X, d)$ to the moduli space of vortices $\overline{M}_n^K(C, X, d)$.*

Proof. That the map Z is a bijection follows from Mundet's Theorem 5.9 applied to the principal component. We check that the map is a homeomorphism. The topology on the coarse moduli space $\overline{\mathcal{M}}_n^G(C, X, d)$ is induced from specialization in families: For any convergent sequence $[(P_\nu, u_\nu)] \rightarrow [(P, u)]$ there exists an analytic family \hat{C} of nodal curves over a connected complex manifold S , a family of holomorphic G -bundles $P \rightarrow C \times S$, a family of maps $\hat{C} \rightarrow P(X)$, and a convergent sequence $s_\nu \in S$ such that (P_ν, u_ν) resp. (P, u) is isomorphic to the fiber over s_ν resp. s . Fixing a reduction of structure group to K and using the correspondence between holomorphic structures and connections gives a family $(A_s \in \mathcal{A}(P), u_s : \hat{C}_s \rightarrow P(X))$ of connections and sections on a fixed K -bundle P . If $s_\nu \in S$ is a sequence converging to $s \in S$ as $\nu \rightarrow \infty$, then $A_{s_\nu} \rightarrow A$ uniformly in all derivatives and u_{s_ν} Gromov converges to u_s . In particular, the principal component $u_{s_\nu,0}$ converges to $u_{s,0}$ uniformly in all derivatives on compact subsets of the complement of the bubbling set. Then $u_{s_\nu}^* P(\Phi) \rightarrow u_s^* P(\Phi)$ in the L^p topology and uniformly on compact subsets of the complement of the bubbling set, and so $F_{A_{s_\nu} + u_{s_\nu}^* P(\Phi)\omega_C} \rightarrow F_{A_s + u_s^* P(\Phi)\omega_C}$ in the L^p -topology on $\Omega^2(C, P(\mathfrak{k}))$ and uniformly on compact subsets of the complement of the bubbling set. Let $\xi_{A,u}$ denote the unique global minimum of $\mathcal{I}(P)$, so that the correspondence is given by $(A, u) \mapsto \exp(i\xi_{A,u})(A, u)$. The $F_{\exp(i\xi_{A,u})(A_s, u_s)}$ converges to $F_{\exp(i\xi_{A,u})(A, u)} = 0$ in L^p . By the implicit function theorem, there exists a unique complex gauge transformation of the form $\exp(i\xi'_\nu)$ such that $\exp(i\xi'_\nu)\exp(i\xi_{A,u})(A_\nu, u_\nu)$ is a

vortex, with $\xi'_\nu \rightarrow 0$ in $W^{1,p}$. Since $\exp(i\xi'_\nu)\exp(i\xi_{A,u}) = \exp(i\xi_{A_\nu,u_\nu}) \bmod \mathcal{K}(P)$, this implies $\xi_{A_s,u_s} \rightarrow \xi_{A,u}$ in $W^{1,p}$. In particular, for $p > 2$ this implies $\xi_{A_s,u_s} \rightarrow \xi_{A,u}$ in C^0 , which implies that Z is continuous. Continuity of the inverse map $\overline{M}_n^K(C, X) \rightarrow \overline{M}_n^G(C, X)$ follows from the fact that $\overline{M}_n^G(C, X)$ is a coarse moduli space for C^0 families of gauged maps. This in turn follows from its construction via Quot scheme methods as in Section 4.5. Namely, for each bundle one finds a point in the Grassmannian corresponding to a realization of the bundle as a quotient; the construction of this point depends continuously on the connection and curve chosen. \square

Lemma 5.15. $\overline{\mathcal{M}}_n^{\text{pre},G}(C, X)$ is an Artin stack equipped with a morphism $\overline{\mathcal{M}}_n^{\text{pre},G}(C, X) \rightarrow \overline{\mathcal{M}}_n^G(C, X)$ collapsing unstable components.

Proof. $\overline{\mathcal{M}}_n^{\text{pre},G}(C, X)$ is the pre-image of $\overline{\mathcal{M}}_n^G(C, X)$ under the morphism of Lemma 5.3. \square

The assignment $X \rightarrow \overline{\mathcal{M}}_n^G(C, X)$ is functorial in the following sense, generalizing functoriality of the stacks of stable map in Behrend-Manin [10].

Definition 5.16. The category of *smooth polarized varieties with reductive group actions* has

- (a) (Objects) are data (G, X, L) consisting of a reductive group G , a smooth polarized G -variety X , and an ample G -line bundle $L \rightarrow X$;
- (b) (Morphisms) from (G_0, X_0, L_0) to (G_1, X_1, L_1) consist of pairs of a morphism $\varphi : X_0 \rightarrow X_1$ a *surjective* homomorphism $\psi : G_0 \rightarrow G_1$ and an *injective* homomorphism $\iota : G_1 \rightarrow G_0$ such that φ preserves Hilbert-Mumford weights, that is, if x_0 is fixed by one-parameter subgroup $\mathbb{C}^\times \rightarrow \iota(G_1)$ then x_1 has the same weight as $\varphi(x_0)$.

Remark 5.17. The definition of morphism implies that G_0 is a product of G_1 with the kernel of ψ , and that the semistable locus in X_0 maps to the semistable locus in X_1 .

Proposition 5.18. $X \mapsto \overline{\mathcal{M}}_n^G(C, X)$ extends to a functor from the category of smooth polarized varieties with reductive group actions to (Artin stacks, equivalence classes of morphisms of Artin stacks).

Proof. Consider the composition $\overline{\mathcal{M}}_n^{G_0}(C, X_0) \rightarrow \overline{\mathcal{M}}_n^{G_1}(C, X_1)$. Let (P_0, u_0) be an object of $\overline{\mathcal{M}}_n^{G_0}(C, X_0)$. Any parabolic reduction of $P_0 \times_{G_0} G_1$ to a parabolic subgroup R_1 defines a parabolic reduction of P_1 to $R_0 = \psi^{-1}(R_1)$, via the isomorphism $P \times_{G_0} G_1/R_1 \rightarrow P/R_0$, and the associated graded bundles $\text{Gr}(P_0)$. Any character of the center of R_1 defines a character of the center of R_0 . The image of the associated graded section $\text{Gr}(u_0) : C \rightarrow P(X_0)$ is the associated graded section of the image of u_0 under $P(X_0) \rightarrow P(X_1)$. Since the Hilbert-Mumford weights are preserved, the Mundet weight is the same and the image of the Mundet semistable locus $\overline{\mathcal{M}}_n^{G_0}(C, X_0)$ lies in $\overline{\mathcal{M}}_n^{G_1, \text{pre}}(C, X_1)$. By restriction we obtain a morphism from $\overline{\mathcal{M}}_n^{G_0}(C, X_0)$ to $\overline{\mathcal{M}}_n^{G_1, \text{pre}}(C, X_1)$, and by composition with the collapse map, to $\overline{\mathcal{M}}_n^{G_1}(C, X_1)$. The functor axioms (identity, composition) are immediate from the definition of the collapse maps. \square

In particular taking X_1 and G_1 in the lemma above to be trivial gives:

Corollary 5.19. *There exists a forgetful morphism*

$$f : \overline{\mathcal{M}}_n^G(C, X) \rightarrow \overline{\mathcal{M}}_n(C)$$

which maps $(\hat{C}, P, u, \underline{z})$ to the stable map to C obtained from $(C, \pi \circ u, \underline{z})$ by composing with the projection $C \times X/G \rightarrow C$ and collapsing unstable components as in (20).

In order to investigate splitting properties of the gauged Gromov-Witten invariants we introduce moduli spaces whose combinatorial type is a rooted forest (finite collection of trees) Γ . Denote by $\overline{\mathfrak{M}}_{n,\Gamma}^G(C, X)$, resp. $\overline{\mathfrak{M}}_{n,\Gamma}^{G,st}(C, X)$, resp. $\overline{\mathcal{M}}_{n,\Gamma}^{G,pre}(C, X)$, resp. $\overline{\mathcal{M}}_{n,\Gamma}^G(C, X)$ the stacks of nodal gauged maps resp. nodal gauged maps with stable sections resp. Mundet semistable maps resp. Mundet semistable maps with stable sections of combinatorial type Γ defined as follows:

Definition 5.20. (Stacks of gauged maps with disconnected combinatorial type) Suppose that $\Gamma = \Gamma_0 \cup \Gamma_1 \dots \cup \Gamma_l$ is a disjoint union of trees $\Gamma_0, \dots, \Gamma_l$ equipped with a root vertex $v_0 \in \text{Vert}(\Gamma_0)$ and a homology class $d = d_0 + \dots + d_l \in H_2^G(X, \mathbb{Z})$. Let $\overline{\mathcal{M}}_{n_0, \Gamma_0}^G(C, X)$ be defined as above and for $i \geq 1$ (not containing the root vertex) let

$$\overline{\mathcal{M}}_{n_i, \Gamma_i}^G(C, X, d_i) := \overline{\mathcal{M}}_{0, n_i, \Gamma_i}(X, d_i)/G$$

(the quotient of the moduli stack of parametrized stable maps by the G -action). Let $\overline{\mathcal{M}}_{n,\Gamma}^G(C, X, d)$ be the product of moduli stacks $\overline{\mathcal{M}}_{n_i, \Gamma_i}^G(C, X, d_i)$.

Let $<$ denote the partial ordering on combinatorial types, so that $\Gamma < \Gamma'$ if Γ' is obtained from Γ by collapsing edges. Denote by $\overline{\mathcal{M}}_{n,\Gamma}(C) = \bigcup_{\Gamma' \leq \Gamma} \overline{\mathcal{M}}_{n,\Gamma'}(C)$ the “compactified” stack of nodal curves of combinatorial type Γ and $\overline{\mathcal{C}}_{n,\Gamma}(C) \rightarrow \overline{\mathcal{M}}_{n,\Gamma}(C)$ the universal curve. These stacks of various combinatorial types are related as follows, in the language of tree morphisms [10].

Proposition 5.21. *Any morphism of rooted trees $\Upsilon : \Gamma \rightarrow \Gamma'$ induces a morphism of moduli spaces of nodal resp. stable gauged maps*

$$\overline{\mathfrak{M}}(\Upsilon, X) : \overline{\mathfrak{M}}_{n,\Gamma}^G(C) \rightarrow \overline{\mathfrak{M}}_{n,\Gamma'}^G(C), \quad \overline{\mathcal{M}}(\Upsilon, X) : \overline{\mathcal{M}}_{n,\Gamma}^G(C) \rightarrow \overline{\mathcal{M}}_{n,\Gamma'}^G(C).$$

In particular,

- (a) (Cutting an edge) *If $\Upsilon : \Gamma \rightarrow \Gamma'$ is a morphism cutting an edge, then $\overline{\mathcal{M}}_{n,\Gamma}^G(C, X)$ may be identified with the fiber product $\overline{\mathcal{M}}_{n,\Gamma'}^G(C, X) \times_{(X/G)^2} (X/G)$ over the diagonal $\Delta : (X/G) \rightarrow (X/G)^2$ and $\overline{\mathcal{M}}(\Upsilon, X)$ is projection of the fiber product on the first factor.*
- (b) (Collapsing an edge) *If Γ' is obtained from Γ by collapsing an edge then $\overline{\mathcal{M}}_{n,\Gamma}^G(C, X)$ is isomorphic to $\overline{\mathcal{M}}_{n,\Gamma'}^G(C, X) \times_{\overline{\mathfrak{M}}_{n,\Gamma'}^G(C)} \overline{\mathfrak{M}}_{n,\Gamma}^G(C)$ and $\overline{\mathcal{M}}(\Upsilon, X)$ is projection on the first factor.*

5.3. Toric quotients and quasimaps. In this section we treat the case that X is a vector space equipped with a linear action of a torus G .

Remark 5.22. (Quotients of vector spaces by tori) Suppose X has weights $\mu_1, \dots, \mu_k \in \mathfrak{g}^\vee$. A moment map for the G -action on X is given by

$$(z_1, \dots, z_k) \mapsto \nu - \left(\sum_{i=1}^k \mu_i |z_i|^2 / 2 \right)$$

where $\nu \in \mathfrak{g}^\vee$ is a constant. Assuming ν is rational, the choice of this constant determines a polarization $\mathcal{O}_X(1) \rightarrow X$ given by twisting the trivial bundle with the rational character corresponding to ν . The semistable locus is then

$$(29) \quad X^{\text{ss}} = \{(z_1, \dots, z_k) \mid \text{span}\{\mu_i, z_i \neq 0\} \ni \nu\}.$$

The git quotient $X//G$ is then a toric variety with residual action of the torus $(\mathbb{C}^\times)^k/G$. The git quotient $X//G$ is proper if the weights μ_1, \dots, μ_k are contained in an open half-space in the real part, and stable=semistable if $\mu_i(\nu) \neq 0$ for all i . Note that $X//G$ depends on the choice of ν . The components of the hyperplanes $\ker \mu_i$ are called *chambers* for ν .

Example 5.23. (The projective plane and its blow-up as a quotient of affine four-space) Suppose that $X = \mathbb{C}^4$ and $G = (\mathbb{C}^\times)^2$ acting with weights $(-1, 0), (-1, 0), (-1, -1), (0, -1)$.

- (a) For $\nu = (1, 2)$ the unstable locus has a component given by the sum of the weight spaces with weights $(-1, 0), (-1, -1)$ and a component equal to the weight space with weight $(0, -1)$. The quotient $X//G$ is isomorphic to \mathbb{P}^2 via the map

$$[x_1, x_2, x_3, x_4] = [(1, x_4^{-1})(x_1, x_2, x_3, x_4)] = [(x_1, x_2, x_3 x_4^{-1}, 1)] \mapsto [x_1, x_2, x_3 x_4^{-1}] \in \mathbb{P}^2.$$

- (b) For $\nu = (2, 1)$, the semistable locus has a component given by the sum of the weight spaces with weights $(0, -1), (-1, -1)$ and a component weight $(0, -1)$. The quotient $X//G$ is isomorphic to the blow-up of \mathbb{P}^2 with the map to \mathbb{P}^2 blowing down the exceptional divisor given by $[x_1, x_2, x_3, x_4] \mapsto [x_1, x_2, x_3 x_4^{-1}]$.

See Figure 15.

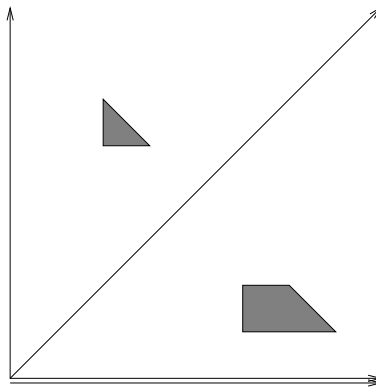


FIGURE 15. Quotients for the $(\mathbb{C}^\times)^2$ action on \mathbb{C}^4

Morphisms from a curve C to the git quotient $X//G$ are closely related to objects in the stack of quasimaps $H^0(C, P \times_G X)/G$ as follows. If $u \in H^0(C, P \times_G X)$ takes values in the

semistable locus then it defines a map to $X//G$, and any map to $X//G$ arises in this way. If C has genus zero, $P \rightarrow C$ has $c_1(P) = d$ and X_j denotes the weight space with weight μ_j then there is an isomorphism of G -modules

$$(30) \quad H^0(C, P \times_G X) \rightarrow X(d) := \bigoplus_j X_j^{\oplus \max(0, (d, \mu_j) + 1)}.$$

Any polarization $\mathcal{O}_X(1)$ of X induces a polarization $\mathcal{O}_{X(d)}(1)$ by taking the moment map resp. polarization to be given by $\nu \in \mathfrak{g}^\vee$. We say that a quasimap $u \in H^0(C, P \times_G X)$ is *(semi)stable* if it is (semi)stable for the polarization $\mathcal{O}_{X(d)}(1)$.

Proposition 5.24. *For any $d \in H_2^G(X, \mathbb{Z})$, there exists a constant ρ_0 such that if stable=semistable for the G -action on $X(d)$ and $\rho > \rho_0$ then a gauged map $(P, A, u) \in \mathcal{H}(P, X)$ of class d is ρ -semistable iff $u \in H^0(C, P \times_G X)$ is semistable for the action of G , so that there is an isomorphism of stacks*

$$\mathcal{M}^G(C, X, d) \cong H^0(C, P \times_G X) // G = X(d) // G.$$

Proof. Since G is abelian, there are no parabolic reductions and Mundet's criterion for semistability becomes

$$\mu(\sigma, \lambda) = \int_{[C]} (c_1(P) + \rho P(\Phi) \circ \text{Gr}(u)_0[\omega_C], -\lambda) \leq 0$$

where λ represents an infinitesimal automorphism of the bundle P , that is, an element of the group G . For ρ sufficiently large, we may ignore the term involving $c_1(P)$ and obtain the stability condition for the action of G on $H^0(C, P \times_G X)$. \square

Example 5.25. (a) (Projective Space) Let $X = \mathbb{C}^k$ with $G = \mathbb{C}^\times$ acting diagonally. Identify $H_2^G(X, \mathbb{Z}) \cong \mathbb{Z}$. Then $X(d) = \mathbb{C}^{kd}$ and $\mathcal{M}^G(C, X, d) = \mathbb{C}^{kd} // \mathbb{C}^\times = \mathbb{P}^{kd-1}$. A polynomial $[u] \in \mathcal{M}^G(C, X, d)$ defines a map to \mathbb{P}^{k-1} of degree d iff its components have no common zeroes.

(b) (The projective plane and its blow-up as a quotient by a two-torus) Suppose that $X = \mathbb{C}^4$ and $G = (\mathbb{C}^\times)^2$ acting with weights $(1, 0), (1, 0), (1, 1), (0, 1)$. With $d = (1, 0)$, we have $X(d) = \mathbb{C}_{(0,1)} \oplus \mathbb{C}_{(1,1)}^{\oplus 2} \oplus \mathbb{C}_{(1,0)}^{\oplus 4}$. The moduli spaces of gauged maps are \mathbb{P}^5 for $\nu = (1, 2)$ or $\text{Bl}_{\mathbb{P}^1}(\mathbb{P}^5)$ for $\nu = (2, 1)$. For example, by Thaddeus [90] the two quotients are related by blow-up along $\mathbb{C}_{(1,1)}^{\oplus 2} // G = \mathbb{P}^1$.

The comparison between the vortex equations and quasimaps has been investigated from the symplectic point of view by J. Wehrheim [95], based on earlier work of Cieliebak-Salamon [15]. The space of quasimaps appears in the work of Morrison-Plesser [64], Givental [31], Lian-Liu-Yau [55] etc. on mirror symmetry as an algebraic model for the space of stable maps to the quotient $X//G$.

5.4. Affine gauged maps. Let X be a G -variety as above. In this section we construct the stack $\overline{\mathcal{M}}_{n,1}^G(\mathbb{A}, X)$ of affine gauged maps. These are used later to construct the quantum Kirwan morphism. The following extends Definition 1.2 to the case of orbifold target $X//G$.

Definition 5.26. (Affine gauged maps) An n -marked affine gauged map to X over a scheme S consists of

- (a) (Projective weighted line) a weighted projective line $C = \mathbb{P}[1, r]$ for some $r > 0$
- (b) (Marking) an n -tuple of distinct points $(z_1, \dots, z_n) : S \rightarrow (C - \{\infty\})^n$, where $\infty := B\mu_r$ is the stacky point at infinity;
- (c) (Scaling) a non-zero meromorphic one-form $\lambda \in H^0(C \times S, T_C^\vee(2\infty))$ and
- (d) (Representable morphism) a representable morphism $u : C \times S \rightarrow X/G$ such that $u(\infty, s) \in X//G$ for all $s \in S$.

A *morphism* of n -marked affine gauged maps $(\underline{z}_j, \lambda_j, u_j)$ consists of an automorphism $\psi : C \rightarrow C$ mapping $z_{0,i}$ to $z_{1,i}$ and pulling back λ_1 to λ_0 and an isomorphism of $u_1 \circ \psi$ with u_0 .

The complement of ∞ in C has the structure of an affine line determined by λ , hence the use of the terminology *affine*. The category $\mathcal{M}_{n,1}^G(\mathbb{A}, X)$ of n -marked affine gauged maps to X/G has the structure of an Artin stack: In the case that $X//G$ is a free quotient, it is an open substack of the stack $\mathrm{Hom}_{\overline{\mathfrak{M}}_{n,1}(\mathbb{A})}(\overline{\mathcal{C}}_{n,1}(\mathbb{A}), X/G)$ considered in Example 4.3. More generally, in the case that $X//G$ has orbifold singularities, $\mathcal{M}_{n,1}^G(\mathbb{A}, X)$ is an open substack of $\mathrm{Hom}_{\overline{\mathfrak{M}}_{n,1}^{\mathrm{tw}}(\mathbb{A})}(\overline{\mathcal{C}}_{n,1}^{\mathrm{tw}}(\mathbb{A}), X/G)$ where $\overline{\mathfrak{M}}_{n,1}^{\mathrm{tw}}(\mathbb{A})$ is defined in Definition 4.2.

Definition 5.27. (Nodal affine gauged maps) A *nodal n -marked affine gauged map* to X over a scheme S consists of a nodal marked scaled affine curve $C = (C, \lambda, z_0, \dots, z_n)$ over S , possibly twisted at the nodes of infinite scaling and the root marking, and a representable morphism $u : C \rightarrow X/G$. In addition we require that

- (a) (Root marking is target-stable) $u(z_0) \in I_{X//G}$;
- (b) (Infinite area components are target-stable) on any component such that λ is infinite, u takes values in the stable locus $X//G$;
- (c) (Zero area components are bundle-stable) the bundle is stable, hence trivializable, on the locus on which the scaling is zero.

A *morphism* of affine gauged maps $(C, \lambda, \underline{z}, u : C \rightarrow X/G)$ to $(C', \lambda', \underline{z}', u' : C' \rightarrow X/G)$ is a morphism $\phi : C \rightarrow C'$ of scaled curves from $(C, \lambda, \underline{z})$ to $(C', \lambda', \underline{z}')$ such that $u = u' \circ \phi$. The *homology class* of $u : C \rightarrow X/G$ is $u_*[C] \in H_2^G(X, \mathbb{Q})$ (integral in the absence of orbifold singularities on the curve C). A affine gauged map over S is *stable* if every fiber $u_s : C_s \rightarrow X$ admits only finitely many automorphisms, or equivalently, every component on which u has zero homology class has at least three special points or two special points and a non-trivial scaling.

Denote by $\overline{\mathcal{M}}_{n,1}^G(\mathbb{A}, X, d)$ resp. $\overline{\mathfrak{M}}_{n,1}^G(\mathbb{A}, X, d)$ the stack of stable resp. not-necessarily stable scaled curves of genus zero and homology class γ and by $\overline{\mathcal{M}}_{n,1}^G(\mathbb{A}, X)$ the sum over homology classes.

Theorem 5.28. $\overline{\mathfrak{M}}_{n,1}^G(\mathbb{A}, X, d)$ resp. $\overline{\mathcal{M}}_{n,1}^G(\mathbb{A}, X, d)$ is an Artin stack resp. proper Deligne-Mumford stack.

Proof. It follows from Example 4.3 that the hom-stack $\mathrm{Hom}_{\overline{\mathfrak{M}}_{n,1,\Gamma}^{\mathrm{tw}}(\mathbb{A})}(\overline{\mathcal{C}}_{n,1,\Gamma}^{\mathrm{tw}}(\mathbb{A}), X/G)$ is an Artin stack, since $\overline{\mathcal{C}}_{n,1,\Gamma}^{\mathrm{tw}}(\mathbb{A}) \rightarrow \overline{\mathfrak{M}}_{n,1,\Gamma}^{\mathrm{tw}}(\mathbb{A})$ is proper and X is smooth. The conditions defining $\overline{\mathcal{M}}_{n,1}^G(\mathbb{A}, X/G, d)$ (values in the semistable locus where $\lambda = \infty$) are open and so $\overline{\mathcal{M}}_{n,1}^G(\mathbb{A}, X/G, d)$

is an open substack of $\mathrm{Hom}_{\overline{\mathfrak{M}}_{n,1,\Gamma}^{\mathrm{tw}}(\mathbb{A})}(\overline{\mathfrak{C}}_{n,1,\Gamma}^{\mathrm{tw}}(\mathbb{A}), X/G)$. Furthermore, by assumption G acts freely on the semistable locus in X and so $\overline{\mathcal{M}}_{n,1}^G(\mathbb{A}, X/G, d)$ has finite automorphism groups, and so is Deligne-Mumford. Properness is equivalent to properness of the underlying coarse moduli space by Proposition 4.9. This in turn follows from the compactness Theorem 3.20. \square

Theorem 5.29. *Suppose that X is either a smooth polarized projective G -variety or a polarized vector space with linear action of G and proper moment map. The coarse moduli space of $\overline{\mathcal{M}}_{n,1}^G(\mathbb{A}, X)$ is homeomorphic to the moduli space of affine symplectic vortices $\overline{M}_{n,1}^K(\mathbb{A}, X)$.*

Proof. This is mostly proved in [93] using the heat flow for gauged maps in [92]. We sketch the proof: Any morphism $u : C \rightarrow X/G$ with $u(z_0) \in X//G$ determines, by restriction, a pair (A, u) on $\mathbb{A} \cong C - \{z_0\}$ taking values in the semistable locus, which can be complex-gauge-transformed (using the implicit function theorem) to a pair satisfying the vortex equations outside of a sufficiently large ball. The heat flow for gauged maps provides complex gauge transform to a symplectic vortex; the convexity of Mundet's functional implies that the complex gauge transformation is unique up to unitary gauge transformation. Let $C \rightarrow S, u : C \rightarrow X/G, \lambda, \underline{z} : S \rightarrow C^n$ be a family of stable affine gauged maps and $s_0 \in S$. After restricting to a neighborhood of s_0 we may assume that the bundles are obtained from a fixed principal K -bundle on C_{s_0} and family of connections on C_{s_0} via gluing. For each $s \in S$, there is a unique-up-to-unitary gauge transformation $g_s \in \mathcal{G}(P)$ such that $g_s(A_s, u_s)$ is a vortex, obtained as the minimum of a functional $\tilde{\psi}_s$ obtained by integrating the moment map. We have $F_{g_s(A_s, u_s)} \rightarrow F_{g_s(A_{s_0}, u_{s_0})_j}$ as $s \rightarrow s_0$ in L^p , by convergence away from the bubbling set, for any component $(A_{s_0}, u_{s_0})_j$ of (A_{s_0}, u_{s_0}) with finite scaling. By the implicit function theorem g_s converges to g_{s_0} in a suitable Sobolev $1, p$ -space for $p > 2$, hence in C^0 . Continuity of the inverse map $\overline{M}_{n,1}^K(\mathbb{A}, X) \rightarrow \overline{M}_{n,1}^G(\mathbb{A}, X)$ follows from the fact that $\overline{M}_{n,1}^G(\mathbb{A}, X)$ is a coarse moduli space for C^0 families of gauged maps, by its construction via Quot scheme methods as in Section 4.5. Let (C_s, P_s, A_s, u_s) be a family of nodal affine vortices over a topological space S . (C_s, P_s, A_s) defines a continuous family of holomorphic bundles, denoted (C_s, P_s^C) . Any such bundle is the pull-back of the universal deformation $P^{\mathrm{univ}} \rightarrow C^{\mathrm{univ}}$ of $(C_{s_0}, P_{s_0}^C)$ by some continuous map $S \rightarrow S^{\mathrm{univ}}$, where $P_{s_0}^C$ is the holomorphic bundle defined by A_0 . Consider u_s as a continuous family of holomorphic maps to $P^{\mathrm{univ}}(X)$, with Gromov limit $u_0 : C_0 \rightarrow P^{\mathrm{univ}}(X)$. The latter is also the limit in the algebraic sense of the maps u_s , that is, the limit of the corresponding points $[u_s]$ in the moduli space of stable maps to $P^{\mathrm{univ}}(X)$. Taking the universal deformation of u_0 realizes u_0 as an algebraic specialization of u_s , which shows that that map $\overline{M}_{n,1}^K(\mathbb{A}, X) \rightarrow \overline{M}_{n,1}^G(\mathbb{A}, X)$ is continuous. \square

Following Behrend-Manin [10] in the case of stable maps, we show that the moduli stacks of affine gauged maps are functorial for suitable morphisms of G -varieties.

Proposition 5.30. *There is a canonical morphism $\overline{\mathcal{M}}_{n,1}^{G,\mathrm{pre}}(\mathbb{A}, X) \rightarrow \overline{\mathcal{M}}_{n,1}^G(\mathbb{A}, X)$, given by (recursively) collapsing unstable components.*

Proof. Given a family $u : C \rightarrow X/G$ of affine gauged maps to X and an ample G -line bundle $L \rightarrow X$ define

$$C^{st} = \text{Proj} \bigoplus_{n \geq 0} \pi_* (\omega_{C/S}^\lambda (z_1 + \dots + z_n) \otimes u^* L^3)^{\otimes n}.$$

The map u factors through C^{st} and commutes with base change, in the case that the family arises from forgetting a marking from a stable family by the similar arguments to those in Behrend-Manin [10]. As in the case of (21), it is necessary to perform this construction *twice* in order to produce a stable affine gauged map. The general case reduces to this one, by adding markings locally. The orbifold case is as in [3, Section 9], by taking the proj relative to the target stack. \square

Recall category of *smooth polarized G -varieties* from Definition 5.16.

Corollary 5.31. $X \mapsto \overline{\mathcal{M}}_{n,1}^G(\mathbb{A}, X)$ extends to a functor from the category of smooth polarized G -varieties to Deligne-Mumford stacks.

Proof. Given morphisms $\phi : X_0 \rightarrow X_1, G_0 \rightarrow G_1$ we obtain a morphism from $\overline{\mathcal{M}}_{n,1}^{G_0}(\mathbb{A}, X_0)$ to $\overline{\mathcal{M}}_{n,1}^{G_1, \text{pre}}(\mathbb{A}, X_1)$ by composing u with ϕ . Composing with the collapsing morphism 5.30 gives the required morphism of moduli stacks. \square

Example 5.32. (Affine gauged maps in the toric case) Suppose that G is a torus and X a vector space with weights μ_1, \dots, μ_N . Then $\mathcal{M}_{1,1}^G(\mathbb{A}, X) = \text{Hom}(\mathbb{A}, X)^{\text{ss}}/G$ where $\text{Hom}(\mathbb{A}, X)^{\text{ss}}$ is the space of morphisms from \mathbb{A} to X that are generically semistable, that is, $u : \mathbb{A} \rightarrow X$ such that $u^{-1}(X^{\text{ss}}) \subset \mathbb{A}$ is non-empty.

- (a) (Projective space quotient) If $X = \mathbb{C}^k$ with $G = \mathbb{C}^\times$ acting diagonally, the component of homology class $d \in H_2^G(X, \mathbb{Z}) \cong \mathbb{Z}$ is

$$\text{Hom}(\mathbb{A}, X, d)^{\text{ss}}/G = \left\{ \sum_{e \leq d} (a_{e,1}, \dots, a_{e,k}) z^e \mid (a_{d,1}, \dots, a_{d,k}) \neq 0 \right\} / G$$

For example, if $d = 1$ and $k = 2$ then

$$\mathcal{M}_{1,1}^G(\mathbb{A}, X, 1) = \{(a_{1,1}z_1 + a_{0,1}, a_{1,2}z_2 + a_{0,2}), (a_{1,1}, a_{1,2}) \neq 0\}/G$$

is the total space of $\mathcal{O}_{\mathbb{P}}(1)^{\oplus 2}$. Its boundary is isomorphic to $\overline{\mathcal{M}}_{0,2}(\mathbb{P}, [\mathbb{P}]) \cong \mathbb{P}^2$, the moduli space of twice-marked stable maps of degree $[\mathbb{P}]$, by the map which attaches a trivial affine gauged map at the marking z_1 .

- (b) (Point quotient) The case of $X = \mathbb{C}$ is studied from the point of view of vortices in Jaffe-Taubes [42]. To describe this classification, let $\text{Sym}^d(\mathbb{A}) = \mathbb{A}^d/S_d$ denote the symmetric product. Jaffe-Taubes [42] show that the map

$$\mathcal{M}_{1,1}^G(\mathbb{A}, X, d) \rightarrow \text{Sym}^d(\mathbb{A}), [u] \mapsto u^{-1}(0)$$

is a homeomorphism on coarse moduli spaces, which is obvious from the algebraic description given here.

- (c) (Weighted projective line quotient) The following is an example with orbifold singularities in the quotient $X//G$. Let \mathbb{C}_2 resp. \mathbb{C}_3 denote the weight space for $G_{\mathbb{C}} = \mathbb{C}^{\times}$ with weight 2 resp. 3 so that $X = \mathbb{C}_2 \oplus \mathbb{C}_3$ and $X//G = \mathbb{P}[2, 3]$. Identifying $H_2^G(X, \mathbb{Q}) \cong \mathbb{Q}$ so that $H_2^G(X, \mathbb{Z}) \cong \mathbb{Z}$ we see that for complex numbers $a_0, b_0, a_1, b_1, \dots$

$$\begin{aligned}\mathcal{M}_{1,1}^G(\mathbb{A}, X, 0) &= \{(a_0, b_0) \neq 0\}/G \cong \mathbb{P}[2, 3] \\ \mathcal{M}_{1,1}^G(\mathbb{A}, X, 1/3) &= \{(a_0, b_1 z + b_0), b_1 \neq 0\}/G \cong \mathbb{C}^2/\mathbb{Z}_3 \\ \mathcal{M}_{1,1}^G(\mathbb{A}, X, 1/2) &= \{(a_1 z + a_0, b_1 z + b_0), a_1 \neq 0\}/G \cong \mathbb{C}^3/\mathbb{Z}_2 \\ \mathcal{M}_{1,1}^G(\mathbb{A}, X, 2/3) &= \{(a_1 z + a_0, b_2 z^2 + b_1 z + b_0), b_2 \neq 0\}/G \cong \mathbb{C}^4/\mathbb{Z}_3 \\ \mathcal{M}_{1,1}^G(\mathbb{A}, X, 1) &= \{(a_2 z^2 + a_1 z + a_0, b_3 z^3 + b_2 z^2 + b_1 z + b_0), (a_2, b_3) \neq 0\}/G\end{aligned}$$

The stacks $\overline{\mathcal{M}}_{n,\Gamma}^G(\mathbb{A}, X)$ satisfy functoriality with respect to morphisms of colored trees, similar to Proposition 4.7, with the caveat that because we allow stacky points in the domain, in the case that $X//G$ is only locally free, the gluing maps will not be isomorphisms:

Proposition 5.33. *Any morphism of colored trees $\Upsilon : \Gamma \rightarrow \Gamma'$ induces a morphism of moduli spaces*

$$\overline{\mathfrak{M}}(\Upsilon, X) : \overline{\mathfrak{M}}_{n,1,\Gamma}^G(\mathbb{A}, X) \rightarrow \overline{\mathfrak{M}}_{n,1,\Gamma'}^G(\mathbb{A}, X), \quad \overline{\mathcal{M}}(\Upsilon, X) : \overline{\mathcal{M}}_{n,1,\Gamma}^G(\mathbb{A}, X) \rightarrow \overline{\mathcal{M}}_{n,1,\Gamma'}^G(\mathbb{A}, X)$$

In particular,

- (a) (Cutting an edge or edges with relations) *If $\Upsilon : \Gamma' \rightarrow \Gamma$ is a morphism corresponding to cutting an edge of Γ , then there is a gluing morphism*
- (31)
$$\mathcal{G}(\Upsilon, X) : \overline{\mathcal{M}}_{n,\Gamma'}^G(\mathbb{A}, X) \times_{\overline{I}_{X/G}^{2m}} \overline{I}_{X/G}^m \rightarrow \overline{\mathcal{M}}_{n,\Gamma}^G(\mathbb{A}, X)$$

where m is the number of cut edges and the second morphism is the diagonal

$$\Delta : \overline{I}_{X/G}^m \rightarrow \overline{I}_{X/G}^{2m}$$

which is an isomorphism in the absence of stacky points in the domain, that is, if $X//G$ is a variety, and in general is an isomorphism after passing to finite covers. The morphism $\overline{\mathcal{M}}(\Upsilon, X)$ is given by projection on the first factor.

- (b) (Collapsing an edge) *If Γ' is obtained from Γ by collapsing an edge then there is an isomorphism*

$$\overline{\mathcal{M}}_{n,\Gamma'}^G(\mathbb{A}, X) \times_{\overline{\mathfrak{M}}_{n,\Gamma'}(\mathbb{A})} \overline{\mathfrak{M}}_{n,\Gamma}(\mathbb{A}) \rightarrow \overline{\mathcal{M}}_{n,\Gamma}^G(\mathbb{A}, X)$$

and $\overline{\mathcal{M}}(\Upsilon, X)$ is given by projection on the first factor.

Proof. In the case that $X//G$ is a variety, these claims are immediate from the definitions. In the case that $X//G$ is a Deligne-Mumford stack, the gluing maps are isomorphisms after passing to the stack $\overline{\mathcal{M}}_{n,\Gamma}^{\text{fr},G}(\mathbb{A}, X)$ of stable affine gauged maps with sections at the stacky points. \square

5.5. Scaled gauged maps. In this section, we construction moduli stacks of scaled maps with projective domain. These are used later to relate the gauged graph potential of X with the graph potential of the quotient $X//G$. Let X be a smooth projectively embedded G -variety and C a smooth connected projective curve.

Definition 5.34. A *nodal scaled gauged map* from C to X consists of a twisted nodal scaled n -marked curve $(\hat{C}, \underline{z}, \omega)$ as in Definition 2.44, with orbifold structures only at the nodes with infinite scaling, together with a morphism $\hat{C} \rightarrow C \times X/G$ consisting of a bundle $P \rightarrow \hat{C}$ and a representable morphism $u : \hat{C} \rightarrow C \times P(X)$. Such a map is *stable* iff

- (a) (Finite scaling) if ω is finite on the principal component then u is stable for the large area chamber;
- (b) (Infinite scaling) if ω is infinite on the principal component then C admits a decomposition into not-necessarily-irreducible components $C = C_0 \cup \dots \cup C_r$ where $u_0 = u|_{C_0}$ is an r -marked stable map $C_0 \rightarrow C \times (X//G)$ and $u_i = u|_{C_i} : C_i \rightarrow C \times X/G$ are stable affine gauged maps.

A morphism of scaled Mundet-stable curves $(\hat{C}, \lambda, \underline{z}, u)$ to $(\hat{C}', \lambda', \underline{z}', u')$ is an isomorphism of the underlying scaled curves $\phi : \hat{C} \rightarrow \hat{C}'$ intertwining the scalings, markings, and morphisms. A nodal gauged map is *stable* if it is Mundet stable and has finitely many automorphisms, that is, each non-principal component with non-degenerate scaling resp. degenerate scaling has at least two resp. three special points.

Let $\overline{\mathfrak{M}}_{n,1}^G(C, X)$ denote the stack of nodal Mundet-semistable scaled gauged maps, and $\overline{\mathcal{M}}_{n,1}^G(C, X)$ the stack of semistable scaled gauged maps. $\overline{\mathcal{M}}_{n,1}^G(C, X)$ is the union of stacks $\overline{\mathcal{M}}_{n,1}^G(C, X)_{<\infty}$ consisting of gauged vortices for large area chamber and a scaling on the underlying curve, and a stack consisting of maps from C to $X//G$ and collections of affine maps to X/G :

$$\overline{\mathcal{M}}_{n,1}^G(C, X, d)_\infty := \bigcup_{r, [I_1, \dots, I_r]} \overline{\mathcal{M}}_r(C, X//G) \times_{(X//G)^r} \prod_{j=1}^r \overline{\mathcal{M}}_{i_j,1}^G(\mathbb{A}, X).$$

Note that $\overline{\mathcal{M}}_{n,1}^G(C, X)$ contains $\overline{\mathcal{M}}_n^G(C, X)$ as the zero section.

Proposition 5.35. $\overline{\mathfrak{M}}_{n,1}^G(C, X)$ is an Artin stack. $\overline{\mathcal{M}}_{n,1}^G(C, X)$ is an open substack equipped with a morphism $\rho : \overline{\mathcal{M}}_{n,1}^G(C, X) \rightarrow \overline{\mathcal{M}}_{0,1}(C) \cong \mathbb{P}$ such that there is an isomorphism

$$\overline{\mathcal{M}}_n^G(C, X, d) \rightarrow \rho^{-1}(0)$$

(with stability on the domain given by the large area chamber $\rho \rightarrow 0$) and there is an isomorphism

$$(32) \quad \bigcup_{i_1 + \dots + i_r = n} \mathcal{M}_r^G(C, X//G) \times_{T_{X/G}^r} \prod_{j=1}^r \overline{\mathcal{M}}_{i_j,1}^G(\mathbb{A}, X) \rightarrow \rho^{-1}(\infty).$$

The coarse moduli space of $\overline{\mathcal{M}}_{n,1}^G(C, X)$ is homeomorphic to the moduli space of scaled vortices $\overline{\mathcal{M}}_{n,1}^K(C, X)$.

Proof. That $\overline{\mathfrak{M}}_{n,1}^G(C, X)$ is an Artin stack follows from Example 4.3, since the universal scaled curve is proper over $\overline{\mathfrak{M}}_{n,1}^G(C, X)$ and $\overline{\mathfrak{M}}_{n,1}^G(C, X)$ is the hom-stack of representable morphisms from the universal scaled curve to X/G . To see that $\overline{\mathcal{M}}_{n,1}^G(C, X, d)$ is an Artin substack we must show that $\overline{\mathfrak{M}}_{n,1}^G(C, X, d)_{<\infty} \setminus \overline{\mathcal{M}}_{n,1}^G(C, X, d)_{<\infty}$ is closed in $\overline{\mathfrak{M}}_{n,1}^G(C, X, d)$. Unfortunately we do not know a purely algebraic argument for this. An argument which uses the Hitchin-Kobayashi correspondences above goes as follows: Suppose that $u : \hat{C} \rightarrow C \times X/G$ is a family of scaled maps over a parameter space S with central fiber u_0 an infinite-area gauged map, that is, a map $C_0 \rightarrow X//G$ together with a collection of affine gauged maps $C_i \rightarrow X/G$, $i = 1, \dots, k$, and for $s \neq 0$, the map u_s ρ_s -unstable with $\rho_s \rightarrow \infty$ as $s \rightarrow 0$. In particular the principal component of u_s can be represented as a pair (A_s, v_s) with (A_s, v_s) flowing under the heat flow in Venugopalan [92] to a limit (A'_s, v'_s) that is reducible. Since K acts locally freely on the zero level set $\Phi^{-1}(0)$ there exists a constant $c > 0$ such that

$$\forall x \in X, \dim(K_x) > 0 \implies \|\Phi(x)\| > c.$$

Using the energy-area identity there exist constants c_0, c_1 such that

$$\|\rho_s^{-1} F_{A'_s} + \rho_s(v'_s)^* P(\Phi)\|_{L^2} \geq c_0 + c_1 |\rho_s|$$

which implies the same estimate for (A_s, v_s) . Now $(A_s, u_s) \rightarrow (A_\infty, v_\infty)$ with $v_\infty^* P(\Phi) = 0$ on the principal component implies that $\|F_{A_s}\|_{L^2}$ is bounded, hence $\|v_s^* P(\Phi)\|_{L^2} \rightarrow \infty$. This contradicts $\|v_\infty^* P(\Phi)\| = 0$. Hence u_0 is not in the closure of the unstable locus in $\overline{\mathfrak{M}}_{n,1}^G(C, X)_{<\infty}$.

If every polystable scaled gauged map is stable then $\overline{\mathcal{M}}_{n,1}^G(C, X)$ is Deligne-Mumford. The homeomorphism of the coarse moduli space to the moduli space of vortices is already established for curves with finite scaling or curves with infinite scaling via Mundet's correspondence and its version for affine curves in Theorem 5.29. It remains to show that the homeomorphisms on these subsets glue together to a homeomorphism on the entire space, that is, that the bijection and its inverse are continuous. Let $C \rightarrow S, u : C \rightarrow X/G, \lambda, \underline{z} : S \rightarrow C^n$ be a family of stable scaled gauged maps and $s_0 \in S$. After restricting to a neighborhood of s_0 we may assume that the bundles are obtained by applying the gluing construction to a principal K -bundle on C_{s_0} to family of connections on C_{s_0} . For each $s \in S$, there is a unique-up-to-unitary gauge transformation $g_s \in \mathcal{G}(P)$ such that $g_s(A_s, u_s)$ is a vortex, obtained as the minimum of a Mundet functional. We have $F_{g_{s_0}(A_s, u_s)} \rightarrow F_{g_{s_0}(A_{s_0}, u_{s_0})_j}$ as $s \rightarrow s_0$ in a weighted Lebesgue space L_δ^p for any component $(A_{s_0}, u_{s_0})_j$ of (A_{s_0}, u_{s_0}) with finite scaling. By an argument using the implicit function theorem in [93] g_s converges to g_{s_0} in $W_\delta^{1,p}$, hence in C^0 on the complement of the bubbling set. Note that because the area form is blowing up, the L^p convergence does not hold at the bubbling points and indeed there is not C^0 convergence of the complex gauge transformation on the principal component. A similar discussion holds on any of the bubbles on which the limiting scaling is finite: Namely if $\phi_s : B_{r_s}(0) \rightarrow C$ is a sequence of embeddings of balls of radius $r_s \rightarrow \infty$ such that $\phi_s^*(A_s, u_s)$ converges to an affine vortex (A, u) . Then $\phi_s^* F_{A_s, u_s}$ converges to $F_{A, u}$ in L^p , and this implies that the gauge transformations g_s converge in C^0 on the compact subsets of the affine line. Continuity of the inverse map $\overline{\mathcal{M}}_{n,1}^K(C, X) \rightarrow \overline{\mathcal{M}}_{n,1}^G(C, X)$ follows from the fact that $\overline{\mathcal{M}}_{n,1}^G(C, X)$ is a coarse moduli space for C^0 families of gauged maps, which is similar to the case $C = \mathbb{A}$. \square

6. VIRTUAL FUNDAMENTAL CLASSES

The virtual fundamental class theory of Behrend-Fantechi [9] (which is a version of earlier approach of Li-Tian [54]) constructs a Chow class from a perfect relative obstruction theory. Stacks of representable morphisms to quotient stacks by reductive groups have canonical perfect relative obstruction theories, by the same construction in [9] and a deformation result of Olsson [73]. In this section, we construct virtual fundamental classes for stacks of gauged (resp. gauged affine, gauged scaled) maps.

6.1. Sheaves on stacks. Any Artin stack \mathcal{X} comes equipped with a canonical *structure sheaf* of rings $\mathcal{O}_{\mathcal{X}}$ in any of the standard Grothendieck topologies on \mathcal{X} . A *sheaf* on an Artin stack \mathcal{X} will mean a sheaf of $\mathcal{O}_{\mathcal{X}}$ -modules over the lisse-étale site of \mathcal{X} , see Olsson [71], de Jong et al [20, Chapter 62]. A sheaf E is *coherent* for object of site the pull-back of E is coherent, that is, admits presentations $\mathcal{O}_{\mathcal{X}}^n \rightarrow E|U$ of finite type, and furthermore any such presentation has kernel of finite type. The *derived category of bounded complexes of sheaves with coherent cohomology* $D^b \text{Coh}(\mathcal{X})$ is the subcategory of the derived category of complexes of coherent sheaves with coherent bounded cohomology groups. It is a triangulated category obtained by inverting quasi-isomorphisms in the category of complexes of sheaves with coherent cohomology.

Example 6.1. (Examples of complexes of coherent sheaves on Artin stacks)

- (a) (Equivariant sheaves) If $\mathcal{X} = X/G$ is the quotient stack associated to a group action of a group G on a scheme X , then the category of sheaves on \mathcal{X} is equivalent to the category of G -equivariant sheaves on X , by an argument involving simplicial spaces [52, 12.4.5].
- (b) (Cotangent complex) Any morphism of Artin stacks $f : \mathcal{X} \rightarrow \mathcal{Y}$ defines a *cotangent complex* $L_{\mathcal{X}/\mathcal{Y}} \in D^b \text{Coh}(\mathcal{X})$ satisfying the expected properties [71], for example, if $g : \mathcal{Y} \rightarrow \mathcal{Z}$ is another morphism of Artin stacks then there is a distinguished triangle in $D^b \text{Coh}(\mathcal{X}) \dots \rightarrow L_{\mathcal{X}/\mathcal{Z}} \rightarrow L_{\mathcal{X}/\mathcal{Y}} \rightarrow Lf^*L_{\mathcal{Y}/\mathcal{Z}}[1] \rightarrow \dots$

6.2. Cycles on stacks. The notion of *rational Chow group* $A(\mathcal{X})$ of a Deligne-Mumford stack \mathcal{X} is developed in Vistoli [94], and further improved in Kresch [50]. A *cycle* of dimension k on \mathcal{X} is an element of the free abelian group $Z_k(\mathcal{X})$ generated by all integral closed substacks of dimension k so that the group of cycles is

$$Z(\mathcal{X}) = \bigoplus_k Z_k(\mathcal{X}).$$

A *cycle with rational coefficients* of dimension k is an element of the group $Z_k(\mathcal{X}) \otimes \mathbb{Q}$. The *group of rational equivalences* on cycles of dimension k on \mathcal{X} is

$$W_k(\mathcal{X}) = \bigoplus_{\mathcal{Y}} \mathbb{C}(\mathcal{Y})^*$$

the sum of the spaces of non-zero rational functions on substacks \mathcal{Y} of \mathcal{X} of dimension $k + 1$. Set

$$W(\mathcal{X}) = \bigoplus_k W_k(\mathcal{X}), \quad W(\mathcal{X})_{\mathbb{Q}} = W(\mathcal{X}) \otimes \mathbb{Q}.$$

If X is a scheme, there is a homomorphism $\partial_X : W(X) \rightarrow Z(X)$ that takes a rational function on a subvariety of X to the cycle associated to its Weil divisor. For a stack \mathcal{X} , the functors Z, W define sheaves on the étale site of \mathcal{X} , the maps ∂_X define a morphism of sheaves and a

hence a morphism of spaces of global sections $\partial_{\mathcal{X}} : W(\mathcal{X}) \rightarrow Z(\mathcal{X})$. The *Chow group* is the cokernel

$$A(\mathcal{X}) := \text{coker}(\partial_{\mathcal{X}} : W(\mathcal{X}) \rightarrow Z(\mathcal{X}))$$

and the *rational Chow group* is $A(\mathcal{X})_{\mathbb{Q}} = A(\mathcal{X}) \otimes \mathbb{Q}$.

Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of Deligne-Mumford stacks. If f is flat, then there is a *flat pullback* $f^* : Z(\mathcal{Y}) \rightarrow Z(\mathcal{X})$. If f is proper, then there is a *proper push-forward* $f_* : Z(\mathcal{X})_{\mathbb{Q}} \rightarrow Z(\mathcal{Y})_{\mathbb{Q}}$ given for finite flat morphisms by $f_*[\mathcal{X}'] = \deg(\mathcal{X}'/f(\mathcal{X}'))[f(\mathcal{X}')]$; note that for stacks the degree is a rational number, see Vistoli [94, Section 2]. These maps pass to rational equivalences, so that we obtain maps

$$f^* : A(\mathcal{Y}) \rightarrow A(\mathcal{X}) \quad f \text{ flat}, \quad f_* : A(\mathcal{X})_{\mathbb{Q}} \rightarrow A(\mathcal{Y})_{\mathbb{Q}} \quad f \text{ proper}.$$

If $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a regular local embedding of codimension d and $Z \rightarrow \mathcal{Y}$ is a morphism from a scheme V , then there is a Gysin homomorphism

$$f^! : Z(\mathcal{Y}) \rightarrow A(\mathcal{X} \times_{\mathcal{Y}} V) \quad f \text{ regular local embedding}$$

defined by local intersection products. Vistoli [94, Theorem 3.11] proves that this passes to rational equivalence. The Gysin homomorphisms satisfy the usual functorial properties with respect to proper and flat morphisms: For any fiber diagram

$$\begin{array}{ccc} \mathcal{X}'' & \longrightarrow & \mathcal{Y}'' \\ p \downarrow & & \downarrow q \\ \mathcal{X}' & \longrightarrow & \mathcal{Y}' \\ \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$

where $\mathcal{Y}', \mathcal{Y}''$ are schemes and f is a regular local embedding, (i) if q is proper then $f^! q_* = p_* f^!$ and (ii) if q is flat then $f^! q^* = p^* f^!$.

If $f : X \rightarrow Y$ is a morphism of schemes there is a *bivariant Chow group* $A^{\vee}(X \rightarrow Y)$, whose elements α of degree l associate to any morphism $U \rightarrow Y$ and each class $u \in A_k(U)$ a class, denoted $\alpha \cap u$, in $A_{k-l}(X \times_Y U)$ satisfying compatibility with flat pull-back, proper push-forward, and Gysin homomorphisms for regular local embeddings. The definition of bivariant Chow groups extends to *representable* morphisms of stacks $f : \mathcal{X} \rightarrow \mathcal{Y}$ [94, Section 5], and the action on Chow groups of schemes extends to an action on Chow groups of stacks equipped with morphisms to \mathcal{Y} .

The theory of Gromov-Witten invariants requires bivariant Chow theory for representable morphisms of Artin stacks. As explained by Behrend-Fantechi [9, Section 7]

- Proposition 6.2.** (a) *If $\mathcal{X} \rightarrow \mathcal{Y}$ is a representable morphism of Artin stacks, then there exists a bivariant Chow group $A^{\vee}(\mathcal{X} \rightarrow \mathcal{Y})$.*
 (b) *If $\mathcal{X} \rightarrow \mathcal{Y}$ is a regular local immersion then there exists a canonical element $[f] \in A^{\vee}(\mathcal{X} \rightarrow \mathcal{Y})$ whose action on Chow cycles is denoted $f^!$.*
 (c) *If $\mathcal{X} \rightarrow \mathcal{Y}$ is flat then there is a canonical orientation class $[f] \in A^{\vee}(\mathcal{X} \rightarrow \mathcal{Y})$.*

6.3. The equivariant case. Now we define derived categories and Chow groups for G -stacks. Let \mathcal{X} be a G -stack with multiplication $\mu : G \times \mathcal{X} \rightarrow \mathcal{X}$ and projection on the right factor $\rho : G \times \mathcal{X} \rightarrow \mathcal{X}$. and $F \rightarrow \mathcal{X}$ a sheaf. A G -linearization of F is an isomorphism of sheaves $\phi : \mu^* F \rightarrow \rho^* F$ which is compatible with multiplication in the sense that $(\mu \times \text{Id}_{\mathcal{X}})^* \phi$ is equal to $(\text{Id}_G \times \mu)^* \phi$. A G -sheaf on \mathcal{X} is a sheaf together with a linearization. Any G -sheaf F descends to a sheaf F/G on the quotient stack \mathcal{X}/G , so that the cohomology of F/G is the invariant part of the cohomology of F :

$$H^j(\mathcal{X}/G, F/G) \cong H^j(\mathcal{X}, F)^G, \forall j.$$

In particular, if these are all finite then the Euler characteristics satisfy

$$\chi(\mathcal{X}/G, F/G) = \chi(\mathcal{X}, F)^G.$$

The *equivariant derived category* $D^b \text{Coh}^G(\mathcal{X})$ is the derived category of the quotient stack $D^b \text{Coh}(\mathcal{X}/G)$. In particular, any complex of G -sheaves defines an object in $D^b \text{Coh}^G(\mathcal{X})$. Note that if \mathcal{X}/G is Deligne-Mumford, then $D^b \text{Coh}^G(\mathcal{X})$ is the usual derived category of bounded complexes of coherent sheaves, otherwise one needs more complicated constructions involving Cartesian sheaves [71]. The *equivariant cotangent complex* is the cone $L_{\mathcal{X}}^G := \text{Cone}(L_{\mathcal{X}} \rightarrow \mathfrak{g}^{\vee})$ on the morphism $L_{\mathcal{X}} \rightarrow \mathfrak{g}^{\vee}$ induced by the action of G . By the exact triangle for cotangent complexes, if the action of G on \mathcal{X} is locally free, so that \mathcal{X}/G is again a Deligne-Mumford stack then $L_{\mathcal{X}}^G$ descends to $L_{\mathcal{X}/G}$.

Suppose that G is a reductive group, and \mathcal{X} is a proper Deligne-Mumford stack \mathcal{X} of dimension n equipped with an action of G . The *equivariant Chow groups* $A^G(\mathcal{X})$ are defined by Edidin-Graham (for schemes) [23] and Graber-Pandharipande (for stacks) [35] as follows. Let V be an l -dimensional representation of G such that V has an open subset U on which G acts freely and whose complement has codimension more than $n - i$. Let

$$A_i^G(\mathcal{X}) = A_{i+l-g}(U \times_G \mathcal{X})$$

be the i -th *equivariant Chow group*. By [23, Proposition 1] (for schemes; the argument for Deligne-Mumford stacks is the same) $A_i^G(\mathcal{X})$ is independent of the choice of V and U . It satisfies the following properties

- (a) (Functoriality) If \mathcal{X}, \mathcal{Y} are Deligne-Mumford stacks equipped with actions of G then any G -equivariant morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ induces a map $f_* : A^G(\mathcal{X}) \rightarrow A^G(\mathcal{Y})$.
- (b) (Free actions) If the action of G on \mathcal{X} is locally free then $A^G(\mathcal{X}) \rightarrow A(\mathcal{X}/G)$ is an isomorphism, where \mathcal{X}/G is the quotient stack. (This is [35, Lemma 6] in the case $G = \mathbb{C}^{\times}$).

More generally, Kresch has introduced a notion of Chow groups for Artin stacks [51], so that $A(\mathcal{X}/G)$ is isomorphic to $A^G(\mathcal{X})$, for a not-necessarily-free action of a reductive group G on a Deligne-Mumford stack \mathcal{X} .

6.4. Obstruction theories. Often a stack \mathcal{X} is given (at least locally) as a zero locus of a vector bundle $\mathcal{E} \rightarrow \mathcal{Y}$. In such a case, \mathcal{X} is a complete intersection and so carries a fundamental class. The notion of *perfect obstruction theory* for \mathcal{X} keeps some of this information and is enough to reconstruct a virtual fundamental class for \mathcal{X} .

Definition 6.3. An *obstruction theory* for an Deligne-Mumford stack \mathcal{X} is a pair (E, ϕ) where $E \in \text{Ob}(D^b \text{Coh}(\mathcal{X}))$ is an object in the derived category of coherent sheaves in the étale topology and $\phi : E \rightarrow L_{\mathcal{X}}$ is a morphism in the derived category of coherent sheaves such that

- (a) $h^i(E) = 0, i > 0$;
- (b) $h^0(\phi)$ is an isomorphism;
- (c) $h^{-1}(\phi)$ is surjective.

The rank of E is the *virtual dimension* of \mathcal{X} . A *relative obstruction theory* for a morphism of stacks $f : \mathcal{X} \rightarrow \mathcal{Y}$ is defined similarly, but replacing the cotangent complex $L_{\mathcal{X}}$ with its relative version L_f . An obstruction theory (E, ϕ) is *perfect* if E has amplitude in $[-1, 0]$, that is, non-vanishing cohomology only in degrees 0, -1 . Behrend-Fantechi [9] furthermore assume that there is a *global resolution* of E , that is, a complex of vector bundles $F = [F^{-1} \rightarrow F^0]$ together with an isomorphism of F to E in $D^b \text{Coh}(\mathcal{X})$, but this assumption is removed in Kresch [51].

The following is a criterion for a pair (E, ϕ) to give an obstruction theory:

Definition 6.4. A closed immersion $T \rightarrow \overline{T}$ of schemes is called a *square-zero extension with ideal sheaf J* if J is the ideal sheaf of T in \overline{T} and $J^2 = 0$.

Given a morphism $g : T \rightarrow X$, the homomorphism $g^*L_{\mathcal{X}} \rightarrow L_{T/\overline{T}}$ defines an element $\omega(g) \in \text{Ext}^1(g^*L_{\mathcal{X}}, J)$.

Proposition 6.5. [9, Theorem 4.5] *(E, ϕ) is an obstruction theory iff for any square-zero extension $T \rightarrow \overline{T}$ and $g : T \rightarrow \mathcal{X}$ morphism, the obstruction $\phi^*\omega(g) \in \text{Ext}^1(g^*E, J)$ vanishes iff an extension of g to \overline{T} exists, and if $\phi^*(\omega(g)) = 0$, then the extensions form a torsor under $\text{Ext}^0(g^*E, J) = \text{Hom}(g^*h^0(E), J)$.*

Proposition 6.6 (Properties of obstruction theories). (a) *The cotangent complex $L_{\mathcal{X}}$ itself is an obstruction theory if it admits a global presentation, and is perfect iff \mathcal{X} is a local complete intersection.*

- (b) *If (E_j, ϕ_j) is a perfect obstruction theory for \mathcal{X}_j , $j = 0, 1$, then $(E_0 \boxplus E_1, \phi_0 \boxplus \phi_1)$ is a perfect obstruction theory for $\mathcal{X}_1 \times \mathcal{X}_2$ [9, 5.7].*

There is an equivariant version of obstruction theory in the sense of Behrend-Fantechi [9], given as follows. Let \mathcal{X} be a proper Deligne-Mumford G -stack where G is a reductive group, and let U be a free G -variety as in the definition of the equivariant Chow ring above. Let $L_{\pi} \in \text{Ob}(D^b \text{Coh}(\mathcal{X} \times_G U))$ be the relative cotangent complex for $\pi : \mathcal{X} \times_G U \rightarrow U/G$. An *equivariant obstruction theory* is a pair (E, ϕ) where $E \in \text{Ob}(D^b \text{Coh}(\mathcal{X} \times_G U))$ and ϕ is a morphism in $D^b \text{Coh}(\mathcal{X} \times_G U)$ to L_{π} . We suppose that E admits a global presentation $E^{-1} \rightarrow E^0$.

Example 6.7. (Examples of Obstruction Theories)

- (a) (Regular Embeddings [9]) Suppose that $\iota : \mathcal{X} \rightarrow \mathcal{Y}$ is a regular local embedding of smooth Deligne-Mumford stacks. Let E be the dual to the normal complex $N_{\mathcal{X}/\mathcal{Y}}^{\vee} \rightarrow \iota^*\Omega_{\mathcal{Y}}$. Then E has a natural morphism to \mathcal{X} induced by $\iota^*\Omega_{\mathcal{Y}} \rightarrow \Omega_{\mathcal{X}}$, making E into a perfect obstruction theory for \mathcal{X} . The virtual fundamental class is the class $\iota^![\mathcal{Y}]$ defined in Lemma 6.2.

- (b) (Stacks of morphisms to projective schemes [9]) Let C, X be projective schemes such that C is Gorenstein, and $\mathrm{Hom}(C, X)$ the scheme of morphisms from C to X . Let $u : C \times X \rightarrow X$ be the universal morphism and $p : C \times X \rightarrow C$ the projection. Let

$$E = Rp_*(u^*L_X \otimes \omega) = (Rp_*u^*T_X)^\vee.$$

Then E is an obstruction theory for X , perfect if X is smooth and C is a curve [9, 6.3]. Indeed, by the functorial properties of the cotangent complex there is a homomorphism

$$e : f^*L_X \rightarrow L_{C \times \mathrm{Hom}(C, X)} \rightarrow L_{C \times \mathrm{Hom}(C, X)/C} \cong \pi^*L_{\mathrm{Hom}(C, X)}.$$

Then e induces a homomorphism

$$\phi := \pi_*(e^\vee)^\vee : E^\vee \rightarrow L_X^\vee.$$

To prove that ϕ is an obstruction theory [9, 6.3] let T be an affine scheme, $g : T \rightarrow \mathrm{Hom}(C, X)$ a morphism, $F \rightarrow T$ a coherent sheaf on T , $p : C \times T \rightarrow T$ the projection, and $h : C \times T \rightarrow X$ a morphism. By [9, Lemma 6.1], $\mathrm{Ext}^k(h^*L_X, p^*F) \cong \mathrm{Ext}^k(g^*E, F)$. If \bar{T} is a square zero extension of T with ideal sheaf F , then g extends to $\bar{g} : \bar{T} \rightarrow X$ iff h extends to $\bar{h} : C \times \bar{T} \rightarrow X$ iff $\phi^*\omega(g)$ is zero in $\mathrm{Ext}^1(h^*L_X, p^*F) = \mathrm{Ext}^1(g^*E, F)$. The extensions, if they exist, form a torsor under $\mathrm{Hom}(h^*L_X, p^*F) = \mathrm{Ext}^0(g^*E, F)$. By [9, Theorem 4.5], (E, ϕ) is an obstruction theory.

- (c) (Stacks of morphisms to Artin stacks) The construction of an obstruction theory extends to the case that X is an Artin S -stack and $\mathrm{Hom}_S(C, X)$ is replaced by a Deligne-Mumford substack of the stack of *representable* morphisms $\mathrm{Hom}_S^{\mathrm{rep}}(C, X)$, as long as one can show that $\mathrm{Hom}_S^{\mathrm{rep}}(C, X)$ is also an Artin S -stack and E has amplitude in $[-1, 0]$; see [73, Theorem 1.1] for the extension of basic results about deformation theory of morphisms of schemes to the setting of stacks. That the Hom-stack $\mathrm{Hom}_S(C, X)$ is an Artin stack, if C, X are, is not known in general, but holds as long as $X = Y/G$ is a quotient stack for action of a reductive group G on a projective variety Y by Example 4.3 (d). In this case, if X is smooth then E has amplitude in $-1, 0$; cohomology below degree -1 vanishes since $T(Y/G)$ has amplitude in $0, 1$ while vanishing in degree 1 follows from the assumption that the substack is Deligne-Mumford and $H^1(E) = \mathrm{Ext}^{-1}(E, \mathbb{C})$ is the sheaf of infinitesimal automorphisms [73, Theorem 1.5].
- (d) (Moduli stacks of bundles) Let C be a projective scheme and G a reductive group so that $\mathrm{Hom}(C, BG)$ is the moduli stack of G -torsors on C . By Examples 4.1 (c) and 4.3 (c) $\mathrm{Hom}(C, BG)$ has an obstruction theory with $E = (Rp_*\mathfrak{g}[1])^\vee$ where \mathfrak{g} denotes the trivial sheaf with fiber \mathfrak{g} . If C is a projective curve then this obstruction theory is perfect on the substack of irreducible bundles. In fact $\mathrm{Hom}(C, BG)$ is a smooth Artin stack and the obstruction theory coincides with the cotangent complex [85, 3.6.8].
- (e) (Hom-stacks over stacks) Continuing 4.3 (d) Let \mathcal{X} be a Gorenstein Deligne-Mumford curve over an Artin stack \mathcal{Z} , \mathcal{Y} an Artin stack over \mathcal{Z} and suppose that $\mathrm{Hom}_{\mathcal{Z}}^{\mathrm{rep}}(\mathcal{X}, \mathcal{Y})$ is an Artin stack, and $\mathrm{Hom}_{\mathcal{Z}}^{\mathrm{rep}, 0}(\mathcal{X}, \mathcal{Y})$ the sub-stack of $\mathrm{Hom}_{\mathcal{Z}}^{\mathrm{rep}}(\mathcal{X}, \mathcal{Y})$ with finite automorphism group. The restriction of the relation obstruction theory to $\mathrm{Hom}_{\mathcal{Z}}^{\mathrm{rep}, 0}(\mathcal{X}, \mathcal{Y})$ is perfect.
- (f) (Moduli stacks of gauged maps) In particular, $\overline{\mathcal{M}}_{n, \Gamma}^G(C, X)$ has a relative obstruction theory over $\overline{\mathcal{M}}_{n, \Gamma}(C)$ with complex given by $(Rp_*u^*T(X/G))^\vee$.

- (g) (Moduli stacks of affine gauged maps) $\overline{\mathcal{M}}_{n,\Gamma}^G(\mathbb{A}, X)$ as an open subset of $\mathrm{Hom}_{\overline{\mathfrak{M}}_{n,\Gamma}(\mathbb{A})}^{\mathrm{rep}}(\overline{\mathcal{C}}_{n,\Gamma}(\mathbb{A}), X/G)$ has a relative obstruction theory over $\overline{\mathfrak{M}}_{n,\Gamma}(\mathbb{A})$ with complex given by $(Rp_* u^* T(X/G))^\vee$.
- (h) (Moduli stacks of stable maps, equivariant case) If a group G acts on a smooth projective variety X , then the moduli stack of stable maps $\overline{\mathcal{M}}_{g,n}(X)$ admits an equivariant perfect relative obstruction theory over $\overline{\mathcal{M}}_{g,n}$ as in Graber-Pandharipande [35].

6.5. Definition of the virtual fundamental class. Behrend-Fantechi [9] and Kresch [50] construct for any Deligne-Mumford stack \mathcal{X} has an *intrinsic normal cone* of pure dimension zero $C_{\mathcal{X}}$, defined by patching together the quotients $C_{U/M}/f^*T_M$ for local embeddings $f : U \rightarrow M$. (See [50, Theorem 1] for a correction to the argument in [9].) If (E, ϕ) is a perfect obstruction theory with $E = (E^{-1} \rightarrow E^0)$ then the morphism ϕ induces a morphism of cone stacks $C_{\mathcal{X}} \rightarrow E^{\vee,1}/E^{\vee,0}$. Let C_E denote the fiber product of $E^{\vee,1}$ and $C_{\mathcal{X}}$ over $E^{\vee,1}/E^{\vee,0}$.

Definition 6.8. (Virtual fundamental classes)

- (a) (Non-equivariant case) The *virtual fundamental class* $[\mathcal{X}]$ (depending on (E, ϕ)) is the intersection of C_E with the zero section of $E^{\vee,1}$ in $A(\mathcal{X})$. By [9, 5.3], $[\mathcal{X}]$ is independent of the choice of global resolution used to construct it.
- (b) (Equivariant virtual fundamental classes) In the equivariant case, the morphism $\pi : U \times_G \mathcal{X} \rightarrow U/G$ is of Deligne-Mumford type and gives an intrinsic normal cone $C_{\mathcal{X}} \in A_0^G(\mathcal{X}) = A_0(\mathcal{X} \times_G U)$. One obtains a virtual fundamental class in $A^G(\mathcal{X})$.
- (c) (Relative virtual fundamental classes) Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a representable morphism of algebraic stacks, and $A^\vee(\mathcal{X} \rightarrow \mathcal{Y})$ the bivariant Chow group constructed by Vistoli [94]. If f is flat or a regular immersion, one denotes by $[f] \in A^\vee(\mathcal{X} \rightarrow \mathcal{Y})$ the orientation class of 6.2, and by $f^!$ its action on Chow groups of Deligne-Mumford stacks. Let $[\mathcal{X}] \in A_{\dim(\mathcal{Y})+\mathrm{rk}(E)}(\mathcal{X})$ be the *relative virtual fundamental class* given by intersecting C_E with the zero section of $E^{\vee,1}$ [9], [50].

Example 6.9. (a) (Stable Maps) Let X be a smooth projective variety. The moduli stack $\overline{\mathcal{M}}_{g,n,\Gamma}(X, d)$ of stable maps of combinatorial type Γ is a proper Deligne-Mumford stack equipped with a perfect relative obstruction theory over $\overline{\mathfrak{M}}_{g,n,\Gamma}$ and so has a virtual fundamental class $[\overline{\mathcal{M}}_{g,n,\Gamma}(X, d)]$.

- (b) (Stable gauged maps) Suppose that $\overline{\mathcal{M}}_{\Gamma,n}^G(C, X)$ denotes the closure of the sub-stack of stable pairs of combinatorial type Γ . If $\overline{\mathcal{M}}_{n,\Gamma}^G(C, X)$ is a Deligne-Mumford substack (equivalently in characteristic zero, all automorphism groups are finite) then it has a virtual fundamental class $[\overline{\mathcal{M}}_{n,\Gamma}^G(C, X)] \in A(\overline{\mathcal{M}}_{n,\Gamma}^G(C, X))$, by Example 6.7 (f).

6.6. Compatibility of perfect obstruction theories. As in Behrend-Fantechi [9, p. 51] consider a diagram of Deligne-Mumford stacks

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{u} & \mathcal{X} \\ g \downarrow & & \downarrow f \\ \mathcal{Z}' & \xrightarrow{v} & \mathcal{Z} \end{array}$$

where $v : \mathcal{Z}' \rightarrow \mathcal{Z}$ is a local complete intersection morphism with finite unramified diagonal over a stack \mathcal{Y} . Let $E \rightarrow L_{\mathcal{X}}$ and $F \rightarrow L_{\mathcal{X}'}$ be perfect relative obstruction theories for \mathcal{X} and \mathcal{X}' over \mathcal{Y} , respectively. A *compatibility datum* for E and F is a triple of morphisms in $D(\mathcal{O}_{\mathcal{X}'})$ giving rise to a morphism of distinguished triangles

$$\begin{array}{ccccccc} u^*E & \xrightarrow{\phi} & F & \xrightarrow{\psi} & g^*L_{\mathcal{Z}'/\mathcal{Z}} & \xrightarrow{\chi} & u^*E[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ u^*L_{\mathcal{X}/\mathcal{Y}} & \longrightarrow & L_{\mathcal{X}'/\mathcal{Y}} & \longrightarrow & L_{\mathcal{X}'/\mathcal{X}} & \longrightarrow & u^*L_{\mathcal{X}/\mathcal{Y}}[1] \end{array}$$

We say that E, F are *compatible perfect relative obstruction theories* if there exists a compatibility datum. By [9, 7.5] if E, F are compatible perfect relative obstruction theories, and \mathcal{Z}' and \mathcal{Z} as above are smooth then $v^![\mathcal{X}] = [\mathcal{X}']$.

Example 6.10. (Cutting an edge for stable maps) Let $\Upsilon : \Gamma' \rightarrow \Gamma$ be a morphism of graphs disconnecting an edge, that is, replacing an edge in Γ with a pair of semi-infinite edges in Γ' . We have a morphism of stacks of stable curves $\overline{\mathfrak{M}}(\Upsilon) : \overline{\mathfrak{M}}_{g,n,\Gamma'} \rightarrow \overline{\mathfrak{M}}_{g,n,\Gamma}$ obtained by identifying the two additional markings, and an induced isomorphism $\overline{\mathcal{M}}(\Upsilon) : \overline{\mathcal{M}}_{g,n,\Gamma'} \rightarrow \overline{\mathcal{M}}_{g,n,\Gamma}$, except in the case that there exists an automorphism of a curve of combinatorial type Γ' interchanging the two markings, in which case it is a double cover.

The stack of stable maps $\overline{\mathcal{M}}_{g,n,\Gamma}(X)$ may be identified with the sub-stack of $\overline{\mathcal{M}}_{g,n,\Gamma'}(X)$ consisting of objects with $u(z_{n+1}) = u(z_{n+2})$, where z_{n+1}, z_{n+2} are the new markings. That is, we have a Cartesian diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,n,\Gamma}(X) & \longrightarrow & \overline{\mathcal{M}}_{g,n,\Gamma'}(X) \\ \downarrow & & \downarrow \\ \overline{\mathfrak{M}}_{g,n,\Gamma} \times X & \xrightarrow{\Delta} & \overline{\mathfrak{M}}_{g,n,\Gamma'} \times X \times X \end{array}$$

where Δ is the diagonal embedding of X . As explained in Behrend [8, p.8] for the case of stable maps, the two perfect relative obstruction theories are compatible which implies

$$[\overline{\mathcal{M}}_{g,n,\Gamma'}(X)] = \Delta^![\overline{\mathcal{M}}_{g,n,\Gamma}(X)].$$

Indeed if Γ' is obtained from Γ by cutting an edge then we check that the obstruction theories are compatible over Δ . Consider the Cartesian diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,n,\Gamma'}(X) & \xrightarrow{\overline{\mathcal{M}}(\Upsilon,X)} & \overline{\mathcal{M}}_{g,n,\Gamma}(X) \\ \downarrow \Psi & & \downarrow \\ \overline{\mathfrak{M}}_{g,n} \times X & \longrightarrow & \overline{\mathfrak{M}}_{g,n} \times X \times X \end{array}$$

Let $\pi : \mathcal{C} \rightarrow \overline{\mathcal{M}}_{g,n,\Gamma}(X)$ denote the universal curve, and let $\mathcal{C}'' = \overline{\mathcal{M}}(\Upsilon, X)^*\mathcal{C}'$ the curve over $\overline{\mathcal{M}}_{g,n,\Gamma'}(X)$ obtained by normalizing at the node corresponding to the edge, with $p : \mathcal{C}'' \rightarrow \mathcal{C}$

the projection, and

$$\mathrm{ev}'' : \mathcal{C}'' \rightarrow X, \quad \mathrm{ev} : \mathcal{C} \rightarrow X$$

the universal maps. So \mathcal{C} is obtained from \mathcal{C}' by identifying the two sections x_1, x_2 of \mathcal{C}' , and is equipped with a section x induced from x_1, x_2 . We have a short exact sequence of complexes relating the push-forward on $\mathcal{C}, \mathcal{C}''$,

$$0 \rightarrow \mathrm{ev}^* TX \rightarrow p_* p^* \mathrm{ev}^* TX \rightarrow x_* x^* \mathrm{ev}^* TX \rightarrow 0,$$

and so an exact triangle

$$(33) \quad R\pi_* \mathrm{ev}^* TX \rightarrow R\pi''_* p^* \mathrm{ev}^* TX \rightarrow x^* \mathrm{ev}^* TX \rightarrow R\pi_* \mathrm{ev}^* TX[1].$$

Now if

$$E_\Gamma(X) := (R\pi_* \mathrm{ev}^* TX)^\vee$$

then

$$\overline{\mathfrak{M}}(\Upsilon)^* E_{\Gamma'}(X) = (R\pi''_* \mathrm{ev}''^* TX)^\vee = (R\pi''_* p^* \mathrm{ev}^* TX)^\vee.$$

Moreover,

$$L_\Delta = TX^\vee | (\overline{\mathfrak{M}}_{g,n,\Gamma} \times X), \quad \Psi^* L_\Delta = x^* \mathrm{ev}^* TX^\vee.$$

So we have an exact triangle

$$\Psi^* L_\Delta[-1] \rightarrow \overline{\mathcal{M}}(\Upsilon, X)^* E_{\Gamma'}(X) \rightarrow E_\Gamma(X) \rightarrow \Psi^* L_\Delta.$$

This gives rise to a homomorphism of distinguished triangles

$$\begin{array}{ccccccc} \overline{\mathfrak{M}}(\Upsilon)^* E_{\Gamma'}(X) & \longrightarrow & E_\Gamma(X) & \longrightarrow & \Psi^* L_\Delta & \longrightarrow & \overline{\mathcal{M}}(\Upsilon)^* E_{\Gamma'}(X)[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \overline{\mathfrak{M}}(\Upsilon)^* L_{\overline{\mathcal{M}}_{g,n,\Gamma'}(X)/\overline{\mathfrak{M}}_{g,n,\Gamma'}} & \longrightarrow & L_{\overline{\mathcal{M}}_{g,n,\Gamma}(X)/\overline{\mathfrak{M}}_{g,n,\Gamma}} & \longrightarrow & L_{\overline{\mathcal{M}}(\Upsilon,X)} & \longrightarrow & \overline{\mathcal{M}}(\Upsilon,X)^* L_{\overline{\mathcal{M}}_{g,n,\Gamma'}(X)/\overline{\mathfrak{M}}_{g,n,\Gamma}}[1] \end{array}$$

Example 6.11. (Collapsing an edge for stable maps) Let $\Upsilon : \Gamma' \rightarrow \Gamma$ be a morphism of modular graphs given by collapsing an edge. Associated to Υ are morphisms of Artin resp. Deligne-Mumford stacks

$$\overline{\mathfrak{M}}(\Upsilon) : \overline{\mathfrak{M}}_{g,n,\Gamma'} \rightarrow \overline{\mathfrak{M}}_{g,n,\Gamma}, \quad \overline{\mathcal{M}}(\Upsilon) : \overline{\mathcal{M}}_{g,n,\Gamma'} \rightarrow \overline{\mathcal{M}}_{g,n,\Gamma}.$$

The inclusion of $\overline{\mathfrak{M}}_{g,n,\Gamma}(X)$ to $\overline{\mathfrak{M}}_{g,n,\Gamma'}(X)$ induces an *isomorphism* of perfect relative obstruction theories. As in Behrend [8], the relative obstruction theories for $\overline{\mathcal{M}}_{g,n,\Gamma}(X), \overline{\mathcal{M}}_{g,n,\Gamma'}(X)$

are related by pull-back. Indeed consider the diagram from [8, p. 15]

$$\begin{array}{ccccc}
 \sqcup_{d' \rightarrow d} \overline{\mathcal{M}}_{g,n,\Gamma'}(X, d') & \longrightarrow & \overline{\mathcal{M}}_{g,n,\Gamma'} \times_{\overline{\mathcal{M}}_{g,n,\Gamma}} \overline{\mathcal{M}}_{g,n,\Gamma}(X, d) & \longrightarrow & \overline{\mathcal{M}}_{g,n,\Gamma}(X, d) \\
 \downarrow & & \downarrow & & \downarrow \\
 \overline{\mathfrak{M}}_{g,n,\Gamma'} & \longrightarrow & \overline{\mathcal{M}}_{g,n,\Gamma'} \times_{\overline{\mathcal{M}}_{g,n,\Gamma}} \overline{\mathfrak{M}}_{g,n,\Gamma} & \longrightarrow & \overline{\mathfrak{M}}_{g,n,\Gamma} \\
 & \searrow & \downarrow & & \downarrow \\
 & & \overline{\mathcal{M}}_{g,n,\Gamma'} & \longrightarrow & \overline{\mathcal{M}}_{g,n,\Gamma}
 \end{array}$$

All the squares are Cartesian and it follows as in [8] (see especially [8, Proposition 8], which uses bivariant Chow theory for representable morphisms of Artin stacks) that

$$\overline{\mathcal{M}}(\Upsilon)^! [\overline{\mathfrak{M}}_{g,n,\Gamma}(X, d)] = \overline{\mathcal{F}}(\Upsilon, X)_* \sum_{d' \mapsto'} [\overline{\mathfrak{M}}_{g,n,\Gamma'}(X, d')]$$

where

$$\overline{\mathcal{F}}(\Upsilon, X) : \overline{\mathcal{M}}_{g,n,\Gamma'}(X, d') \rightarrow \overline{\mathcal{M}}_{g,n,\Gamma'} \times_{\overline{\mathcal{M}}_{g,n,\Gamma}} \overline{\mathcal{M}}_{g,n,\Gamma}(X, d)$$

is the identification with the fiber product.

7. GAUGED GROMOV-WITTEN INVARIANTS

The perfect obstruction theories for hom-stacks in the previous section define virtual fundamental classes on the moduli stacks of stable gauged maps. The construction is similar to Behrend's construction" [8] of algebraic Gromov-Witten invariants, the main difference being that the moduli spaces are not necessarily even virtually smooth, so the splitting axiom only holds for Cartier divisors. We review from Gonzalez-Woodward [33] the proof that these invariants define a trace on the CohFT algebra given by Givental's equivariant Gromov-Witten theory. We also review the construction of Abramovich-Graber-Vistoli [1] of Gromov-Witten invariants for orbifolds, which are needed to construct the graph potential of the git quotient in the case that the quotient is only locally free.

7.1. Equivariant Gromov-Witten theory for smooth varieties. First we explain the construction of equivariant Gromov-Witten invariants for a smooth projective target using the Behrend-Fantechi machinery [9], as explained in Graber-Pandharipande [35]. We adopt the perspective on the splitting axiom adopted in Behrend [8]: invariants are defined for any possibly disconnected combinatorial type, and the splitting axiom can be broken down into *cutting edges* and *collapsing edges* axiom.

Recall that a map $f : C \rightarrow X$ from a projective nodal curve C to a projective G -scheme X is *stable* if it has only finitely many automorphisms, that is, automorphisms of C which preserve

f under composition. For class $d \in H_2(X, \mathbb{Z})$ let $\overline{\mathcal{M}}_{g,n}(X, d)$ denote the moduli stack of n -pointed, homology class d , genus g stable maps to X , as in Example 4.3 (b). By Behrend-Manin [10], $\overline{\mathcal{M}}_{g,n}(X, d)$ is a Deligne-Mumford stack equipped with evaluation maps

$$\text{ev} = (\text{ev}_1, \dots, \text{ev}_n) : \overline{\mathcal{M}}_{g,n}(X, d) \rightarrow X^n$$

and for $2g + n \geq 3$ a *forgetful morphism*

$$f : \overline{\mathcal{M}}_{g,n}(X, d) \rightarrow \overline{\mathcal{M}}_{g,n}$$

obtained by forgetting the stable map and collapsing any unstable components. Furthermore, for any $c > 0$ the union of components $\overline{\mathcal{M}}_{g,n}(X, d)$ with $(d, [\omega]) < c$ is proper. The G -scheme structure on X induces a (strict) G -stack structure on $\overline{\mathcal{M}}_{g,n}(X, d)$.

The stack $\overline{\mathcal{M}}_{g,n}(X, d)$ has a perfect relative obstruction theory over $\overline{\mathfrak{M}}_{g,n}$ whose complex is the dual of the derived push-forward of the pull-back of the tangent sheaf

$$Rp_* u^* TX \in \text{Ob}(D^b \text{Coh}_G(\overline{\mathcal{M}}_{g,n}(X, d)))$$

where p, u are the universal curve and evaluation map

$$\begin{array}{ccc} \overline{\mathcal{C}}_{g,n}(X) \times_G U & \xrightarrow{u} & X \times_G U \\ \downarrow p & & \\ \overline{\mathcal{M}}_{g,n}(X) \times_G U & & \end{array},$$

and TX the equivariant tangent bundle, that is, the relative tangent bundle for the morphism $X \times_G U \rightarrow U/G$. Hence one obtains a virtual fundamental class of expected dimension $[\overline{\mathcal{M}}_{g,n}(X, d)] \in A^G(\overline{\mathcal{M}}_{g,n}(X, d))$. More generally, given any (possibly disconnected) modular graph Γ a similar construction give virtual fundamental classes

$$[\overline{\mathcal{M}}_{g,n,\Gamma}(X, d)] \in A^G(\overline{\mathcal{M}}_{g,n,\Gamma}(X, d)).$$

These satisfy the following properties proved by Behrend [8], continuing Examples 6.10, 6.11:

- Proposition 7.1.** (a) (Constant maps) *If $d = 0$ then $[\overline{\mathcal{M}}_{g,n,\Gamma}(X, d)]$ is obtained by cap product of $[X \times \overline{\mathcal{M}}_{g,n,\Gamma}]$ with the Euler class of $Rp_* u^* TX$.*
 (b) (Products) *If X_0, X_1 are G -varieties then $[\overline{\mathcal{M}}_{g,n,\Gamma}(X_0 \times X_1, (d_0, d_1))] = [\overline{\mathcal{M}}_{g,n,\Gamma}(X_0, d_0)] \times [\overline{\mathcal{M}}_{g,n,\Gamma}(X_1, d_1)]$*
 (c) (Cutting edges) *If $\Upsilon : \Gamma \rightarrow \Gamma'$ is a morphism of modular graphs of type cutting an edge then*

$$[\overline{\mathcal{M}}_{g,n,\Gamma'}(X, d')] = \Delta^! [\overline{\mathcal{M}}_{g,n,\Gamma}(X, d)]$$

where $\Delta : X \rightarrow X \times X$ is the diagonal.

- (d) (Collapsing edges) *If Υ is a morphism of graphs of type collapsing an edge then*

$$\mathcal{M}(\Upsilon)^! [\overline{\mathcal{M}}_{g,n,\Gamma'}(X, d')] = \overline{\mathcal{F}}(\Upsilon, X)_* \sum_{d \rightarrow d'} [\overline{\mathcal{M}}_{g,n,\Gamma}(X, d)]$$

where

$$\overline{\mathcal{F}}(\Upsilon, X) : \overline{\mathcal{M}}_{g,n,\Gamma'}(X, d') \rightarrow \overline{\mathcal{M}}_{g,n,\Gamma'} \times_{\overline{\mathcal{M}}_{g,n,\Gamma}} \overline{\mathcal{M}}_{g,n,\Gamma}(X, d).$$

(e) (Forgetting Tails) *If $\Upsilon : \Gamma \rightarrow \Gamma'$ is a morphism of graphs of type forgetting a tail then*

$$\overline{\mathcal{M}}(\Upsilon, X)^! [\overline{\mathcal{M}}_{g,n,\Gamma'}(X, d)] = [\overline{\mathcal{M}}_{g,n,\Gamma}(X, d)].$$

We now pass from Chow groups/rings to homology/cohomology with rational coefficients. (One can work with more general theories here, as in Behrend-Manin [10].) For any cohomology classes $\alpha \in H_G(X, \mathbb{Q})^n$ and $\beta \in H(\overline{\mathcal{M}}_{g,n,\Gamma}, \mathbb{Q})$ (if $2g + n \geq 3$) pairing with the virtual fundamental class $[\overline{\mathcal{M}}_{g,n,\Gamma}(X, d)] \in H(\overline{\mathcal{M}}_{g,n,\Gamma}(X, d))$ defines a *Gromov-Witten invariant*

$$\langle \alpha; \beta \rangle_{\Gamma, d} = \int_{[\overline{\mathcal{M}}_{g,n,\Gamma}(X, d)]} \text{ev}^* \alpha \cup f^* \beta \in H(BG).$$

These invariants satisfy axioms for morphisms of modular graphs:

Proposition 7.2. (a) (Cutting edges) *If Γ' is obtained from Γ by cutting an edge then*

$$\langle \alpha; \beta \rangle_{\Gamma, d} = \sum_{i=1}^N \langle \alpha, \delta_i, \delta^i; \overline{\mathcal{M}}(\Upsilon)^* \beta \rangle_{\Gamma', d}$$

where $(\delta_i)_{i=1}^N, (\delta^i)_{i=1}^N$ are dual bases for $H_G(X)$ over $H(BG)$.

(b) (Collapsing Edges) *If $\Upsilon : \Gamma \rightarrow \Gamma'$ is a morphism collapsing an edge then*

$$\langle \alpha; \beta \cup \gamma \rangle_{\Gamma, d} = \sum_{d' \mapsto d} \langle \alpha; \overline{\mathcal{M}}(\Upsilon)^* \beta \rangle_{\Gamma', d'}$$

where $\gamma \in H^2(\overline{\mathcal{M}}_{g,n,\Gamma})$ is the dual class for $\overline{\mathcal{M}}(\Upsilon) : \overline{\mathcal{M}}_{g,n,\Gamma} \rightarrow \overline{\mathcal{M}}_{g,n,\Gamma'}$.

(c) (Forgetting tails) *If $\Upsilon : \Gamma \rightarrow \Gamma'$ is a morphism forgetting a tail then for $\alpha' \in H_G^2(X)$,*

$$\langle \alpha, \alpha'; \overline{\mathcal{M}}(\Upsilon)^* \beta \rangle_{\Gamma, d} = (d, \alpha') \langle \alpha; \beta \rangle_{\Gamma', d'}$$

Proof. By Proposition 7.1, and, for the last item, integration over the fiber of the forgetful map. \square

Definition 7.3. (Novikov field) The Novikov field Λ_X for X is the set of all maps $a : H_2(X) := H_2(X, \mathbb{Z})/\text{torsion} \rightarrow \mathbb{Q}$ such that for every constant c , the set of classes

$$\{d \in H_2(X), \langle [\omega], d \rangle \leq c\}$$

on which a is non-vanishing is finite. The delta function at d is denoted q^d . Addition is defined in the usual way and multiplication is convolution, so that $q^{d_1} q^{d_2} = q^{d_1 + d_2}$.

Define as vector spaces the *quantum cohomology of X*

$$QH_G(X) := H_G(X, \mathbb{Q}) \otimes \Lambda_X.$$

By Proposition 7.2,

Theorem 7.4. [8] *$QH_G(X)$ equipped with the maps*

$$\langle \cdot, \cdot \rangle_g = \sum_{d \in H_2(X)} q^d \langle \cdot, \cdot \rangle_{g, d}$$

forms a cohomological field theory after tensoring with the field of fractions of $H(BG)$.

Restricting to genus zero we obtain a map

$$\mu^n : QH_G(X)^n \times H(\overline{M}_{0,n+1}, \mathbb{Q}) \rightarrow QH_G(X)$$

by

$$(\mu^n(\alpha_1, \dots, \alpha_n; \beta), \alpha_0) = \sum_{d \in H_2(X, \mathbb{Z})} q^d \langle \alpha_0, \dots, \alpha_n; \beta \rangle_{0,d} \in \Lambda_X$$

where $(\ , \)$ denotes the pairing on $QH_G(X)$ induced by cup product and integration over $H_G(X)$. From the point of view of CohFT algebras, passing to the field of fractions is not necessary.

A related collection of invariants is expressed as the integrals over *parametrized* stable maps to X . Let $\text{Hom}(C, X, d) \subset \text{Hom}(C, X)$ denote the subscheme of maps of class $d \in H_2(X, \mathbb{Z})$. Compactifications of $\text{Hom}(C, X, d)$ are provided by so-called *graph spaces*

$$\overline{\mathcal{M}}_n(C, X, d) := \overline{\mathcal{M}}_{g,n}(C \times X, (1, d))$$

of stable maps $u : \hat{C} \rightarrow C \times X$ of degree $(1, d)$. Each stable map $u = (u_C, u_X) : \hat{C} \rightarrow C \times X$ has a single component $\hat{C}_0 \subset \hat{C}$ that maps isomorphically onto C via u_C , with all other components mapping to points. We denote by

$$(34) \quad \text{ev} : \overline{\mathcal{M}}_n(C, X) \rightarrow X^n, \quad \text{ev}_C : \overline{\mathcal{M}}_n(C, X) \rightarrow C^n$$

the evaluation maps followed by projection on the second, resp. first factor. The stacks $\overline{\mathcal{M}}_n(C, X, d)$ have equivariant relatively perfect obstruction theories over $\overline{\mathfrak{M}}_n(C)$ with complex given by $Rp_* u^* TX$, where $p : \overline{\mathcal{C}}_n(C, X) \rightarrow \overline{\mathcal{M}}_n(C, X)$ is the universal curve and $u : \overline{\mathcal{C}}_n(C, X) \rightarrow C$ the evaluation map. For any cohomology classes $\alpha \in H(X, \mathbb{Q})^n$ and $\beta \in H(\overline{\mathcal{M}}_{n,\Gamma}(C), \mathbb{Q})$ pairing with the virtual fundamental class $[\overline{\mathcal{M}}_{n,\Gamma}(C, X, d)] \in H(\overline{\mathcal{M}}_{n,\Gamma}(C, X, d))$ defines a *graph Gromov-Witten invariant*

$$(35) \quad \langle \alpha; \beta \rangle_{C,\Gamma,d} = \int_{[\overline{\mathcal{M}}_{n,\Gamma}(C,X,d)]} \text{ev}^* \alpha \cup f^* \beta \in H(BG).$$

These invariants satisfy axioms for morphisms of *rooted* modular trees:

Proposition 7.5. (a) (Cutting edges) *If Γ' is obtained from Γ by cutting an edge then*

$$\langle \alpha; \beta \rangle_{C,\Gamma,d} = \sum_{i=1}^N \langle \alpha, \delta_i, \delta^i; \overline{\mathcal{M}}(\Upsilon)^* \beta \rangle_{C,\Gamma',d}$$

where δ_i, δ^i are dual bases for $H_G(X)$ over $H(BG)$;

(b) (Collapsing Edges) *If $\Upsilon : \Gamma \rightarrow \Gamma'$ is a morphism collapsing an edge then*

$$\langle \alpha; \beta \cup \gamma \rangle_{C,\Gamma,d} = \sum_{d' \mapsto d} \langle \alpha; \overline{\mathcal{M}}(\Upsilon)^* \beta \rangle_{C,\Gamma',d'}$$

where $\gamma \in H^2(\overline{\mathcal{M}}_{n,\Gamma}(C))$ is the dual class for $\overline{\mathcal{M}}(\Upsilon) : \overline{\mathcal{M}}_{n,\Gamma}(C) \rightarrow \overline{\mathcal{M}}_{n,\Gamma'}(C)$.

(c) (Forgetting tails) *If $\Upsilon : \Gamma \rightarrow \Gamma'$ is a morphism forgetting a tail then for $\alpha' \in H_G^2(X)$,*

$$\langle \alpha, \alpha'; \overline{\mathcal{M}}(\Upsilon)^* \beta \rangle_{C,\Gamma,d} = (d, \alpha') \langle \alpha; \beta \rangle_{C,\Gamma',d'}$$

Proof. The proof is similar to Proposition 7.2 and omitted. □

Define

$$(36) \quad \tau_X^n : QH_G(X)^n \times H(\overline{\mathcal{M}}_n(C)) \rightarrow \Lambda_X^G \otimes H(BG), \quad (\alpha, \beta) \mapsto \sum_d q^d \langle \alpha; \beta \rangle_{C,d}.$$

Theorem 7.6. *The maps $(\tau_X^n)_{n \geq 0}$ define a CohFT trace on the CohFT algebra $QH_G(X)$.*

Proof. By Proposition 7.5 (a) and (b). \square

7.2. Gromov-Witten theory for smooth Deligne-Mumford stacks. We review the orbifold Gromov-Witten theory developed by Abramovich-Graber-Vistoli [1], needed in our case if the geometric invariant theory quotient $X//G$ is an orbifold. For simplicity, we restrict to the case without group action.

Let \mathcal{X} be a proper smooth Deligne-Mumford stack. The moduli space of twisted stable maps $\overline{\mathcal{M}}_{g,n}(\mathcal{X})$ discussed in Section 4.4 has a canonical perfect relative obstruction theory given by $Rp_* u^* T_{\mathcal{X}}$ where p, u are the universal curve and evaluation map

$$\begin{array}{ccc} \overline{\mathcal{C}}_{g,n}(\mathcal{X}) & \xrightarrow{u} & \mathcal{X} \\ \downarrow p & & \\ \overline{\mathcal{M}}_{g,n}(\mathcal{X}) & & \end{array}$$

The moduli stack of twisted stable maps admits evaluation maps

$$\text{ev} : \overline{\mathcal{M}}_{g,n}(\mathcal{X}) \rightarrow \overline{I}_{\mathcal{X}}^n, \quad \overline{\text{ev}} : \overline{\mathcal{M}}_{g,n}(\mathcal{X}) \rightarrow \overline{I}_{\mathcal{X}}^n,$$

where the second is obtained by composing with the involution

$$(37) \quad \overline{I}_{\mathcal{X}} \rightarrow \overline{I}_{\mathcal{X}}$$

induced by the map $\mu_r \rightarrow \mu_r, \zeta \mapsto \zeta^{-1}$. Of course these can be mixed by assigning signs to the marked points. The virtual fundamental classes satisfy splitting axioms for morphisms of modular graphs, in particular, for cutting an edge in which case one of the evaluation maps is taken to be with respect to opposite signs on the pair of marked points created by the cutting. Given a homology class $d \in H_2(\mathcal{X}, \mathbb{Q})$, let $\overline{\mathcal{M}}_{g,n}(\mathcal{X}, d)$ denote the moduli stack of stable maps to \mathcal{X} with class d .

Proposition 7.7. ([1, 5.3.1, 5.3.2]) *The virtual fundamental class $[\overline{\mathcal{M}}_{\Gamma,n}(\mathcal{X}, d)]$ satisfy the splitting axioms for morphisms of modular graphs:*

- (a) (Cutting edges) *If $\Upsilon : \Gamma \rightarrow \Gamma'$ is a morphism of modular graphs of type cutting an edge then*

$$\mathcal{G}(\Upsilon, X)^! [\overline{\mathcal{M}}_{g,n,\Gamma'}(\mathcal{X}, d')] = \Delta^! [\overline{\mathcal{M}}_{g,n,\Gamma}(\mathcal{X}, d)]$$

where $\mathcal{G}(\Upsilon, X)$ is the map of (23) and $\Delta : \overline{I}_{\mathcal{X}} \rightarrow \overline{I}_{\mathcal{X}}^2$ is the diagonal.

- (b) (Collapsing edges) *If Υ is a morphism of graphs of type collapsing an edge then*

$$\mathcal{M}(\Upsilon)^! [\overline{\mathcal{M}}_{g,n,\Gamma'}(\mathcal{X}, d')] = \overline{\mathcal{F}}(\Upsilon, X)_* \sum_{d \rightarrow d'} [\overline{\mathcal{M}}_{g,n,\Gamma}(\mathcal{X}, d)]$$

where

$$\overline{\mathcal{F}}(\Upsilon, X) : \overline{\mathcal{M}}_{g,n,\Gamma'}(X, d') \rightarrow \overline{\mathcal{M}}_{g,n,\Gamma'} \times_{\overline{\mathcal{M}}_{g,n,\Gamma}} \overline{\mathcal{M}}_{g,n,\Gamma}^G(X, d)$$

is the identification with the fiber product.

Orbifold Gromov-Witten invariants are defined by virtual integration of pull-back classes using the evaluation maps above. Define *Gromov-Witten invariants*

$$(38) \quad H(\overline{I}_{\mathcal{X}})^{n_+} \times H(\overline{I}_{\mathcal{X}})^{n_-} \times H(\overline{M}_{g,n}) \rightarrow \mathbb{Q},$$

$$(\alpha_+, \alpha_-, \beta) \mapsto \langle \alpha_+, \alpha_-, \beta \rangle_{\Gamma, d} = \int_{[\overline{\mathcal{M}}_{g,n,\Gamma}(\mathcal{X}, d)]} \text{ev}^* \alpha_+ \cup \overline{\text{ev}}^* \alpha_- \cup f^* \beta.$$

Any copy of the cohomology of $H(I_{\mathcal{X}}) \cong H(\overline{I}_{\mathcal{X}})$ can be put on the right hand side by duality; however in order for the splitting axiom to hold one needs to add a factor of r arising from the fiber of $I_{\mathcal{X}} \rightarrow \overline{I}_{\mathcal{X}}$.

Proposition 7.8. [1, 6.1.4] *The orbifold Gromov-Witten invariants satisfy for the following properties:*

(a) (Duality) for $\alpha_1, \dots, \alpha_n \in H(\overline{I}_{\mathcal{X}}, \mathbb{Q})$ a duality axiom

$$\langle \overline{\alpha}_1, \dots, \overline{\alpha}_n, \beta \rangle_{\Gamma, d} = \langle \alpha_1, \dots, \alpha_n, \beta \rangle_{\Gamma, d}$$

(b) (Collapsing an edge) If $\Upsilon : \Gamma \rightarrow \Gamma'$ is of type collapsing an edge then for any labelling d' of Γ' ,

$$\langle \alpha; \beta \cup \gamma \rangle_{\Gamma', d'} = \sum_{d \rightarrow d'} \langle \alpha; \overline{\mathcal{M}}(\Upsilon)^* \beta \rangle_{\Gamma, d}$$

where γ is the dual class to $\overline{\mathcal{M}}(\Upsilon)$.

(c) (Cutting an edge) If Γ' is obtained from Γ by cutting an edge then

$$\langle \alpha; \beta \rangle_{\Gamma, d} = \sum_k \langle \alpha, \delta_k, \delta^k; \overline{\mathcal{M}}(\Upsilon)^* \beta \rangle_{\Gamma', d}$$

where δ_k, δ^k are dual bases of $H(\overline{I}_{\mathcal{X}})$ with respect to the inner product given by $([\overline{I}_{\mathcal{X}}], r \cdot)$ where $r : I_{\mathcal{X}} \rightarrow \mathbb{Z}_{\geq 0}$ is the order of the isotropy group.

The definition of orbifold Gromov-Witten invariants leads to the definition of orbifold quantum cohomology as follows.

Definition 7.9. (Orbifold quantum cohomology) To each component \mathcal{X}_i of $\overline{I}_{\mathcal{X}}$ is assigned a rational number $\text{age}(\mathcal{X}_i)$ as follows. Let (x, g) be a point in \mathcal{X}_i . The element g acts on $T_x \mathcal{X}$ with eigenvalues $(\alpha_1, \dots, \alpha_n)$ with $n = \dim(\mathcal{X})$. Let r be the order of g and define $s_j \in \{0, \dots, r\}$ by $\alpha_j = \exp(2\pi i s_j / r)$. The *age* is defined by

$$\text{age}(\mathcal{X}_i) = (1/r) \sum_{j=1}^n s_j.$$

Let $\Lambda_{\mathcal{X}} \subset \text{Hom}(H_2(X, \mathbb{Q}), \mathbb{Q})$ denote the Novikov field of linear combinations of formal symbols $q^d, d \in H_2(\mathcal{X}, \mathbb{Q})$ where for each c , only finitely many q^d with $(d, [\omega]) < c$ have non-zero coefficient. Let

$$QH(\mathcal{X}) = H(\overline{I}_{\mathcal{X}}) \otimes \Lambda_{\mathcal{X}}$$

denote the *orbifold quantum cohomology* equipped with the *age grading*

$$QH^\bullet(\mathcal{X}) = \bigoplus_{\mathcal{X}_i \subset I_{\mathcal{X}}} H^{\bullet+2\text{age}(\mathcal{X}_i)}(\mathcal{X}_i) \otimes \Lambda_{\mathcal{X}}.$$

Theorem 7.10. *The orbifold Gromov-Witten invariants define the structure of a CohFT on $QH(\mathcal{X})$, in particular, a CohFT algebra structure on $QH(\mathcal{X})$ and the graph invariants define a trace on $QH(\mathcal{X})$.*

Proof. This follows from the splitting axiom (7.7), and the analogous splitting axiom for the graph invariants whose proof is similar. \square

7.3. Twisted Gromov-Witten invariants. We also describe twisted versions of Gromov-Witten invariants arising from vector bundles on the target, see for example Coates-Givental [17]. Under suitable positivity assumptions, these invariants are equal to the Gromov-Witten invariants of hypersurfaces defined by sections.

Definition 7.11. (Twisting class and twisted Gromov-Witten invariants) Let E be a G -equivariant complex vector bundle over a smooth projective G -variety X . Pull-back under the evaluation map $e : \overline{\mathcal{C}}_{g,n}(X) \rightarrow X$ on the universal curve gives rise to a vector bundle $\text{ev}^* E \rightarrow \overline{\mathcal{C}}_{g,n}(X)$, which we can push down to an *index*

$$\text{Ind}_G(E) := Rp_* e^* E$$

in the derived category of bounded complexes of coherent sheaves on $\overline{\mathcal{M}}_{g,n}(X)$. Since p is a local complete intersection morphism, $\text{Ind}_G(E)$ admits a resolution by vector bundles, see [17, Appendix], and we may consider the equivariant Euler class

$$\epsilon(E) := \text{Eul}_{G \times \mathbb{C}^\times}(\text{Ind}_G(E)) \in H_G(\overline{\mathcal{M}}_{g,n}(X)) \otimes \mathbb{Q}[\zeta, \zeta^{-1}]$$

which is invertible in $H_G(\overline{\mathcal{M}}_{n,\Gamma}(X), \mathbb{Q}) \otimes \mathbb{Q}[\zeta, \zeta^{-1}]$, where ζ is the parameter for the action of \mathbb{C}^\times by scalar multiplication in the fibers. The *twisted equivariant Gromov-Witten invariants* associated to $E \rightarrow X$ and type Γ are the maps

$$(39) \quad H_G(X)^n \times H(\overline{\mathcal{M}}_{g,n,\Gamma}) \rightarrow H_G(\text{pt}, \mathbb{Q}) \otimes \mathbb{Q}[\zeta, \zeta^{-1}]$$

$$\langle \alpha; \beta \rangle_{\Gamma, E, d} = \int_{[\overline{\mathcal{M}}_{n,\Gamma}(C, X)]} \text{ev}^* \alpha \cup f^* \beta \cup \epsilon(E).$$

Proposition 7.12. *The twisted invariants satisfy the properties:*

- (a) (Collapsing an edge) *If $\Upsilon : \Gamma \rightarrow \Gamma'$ is of type collapsing an edge then for any labelling d' of Γ' ,*

$$\langle \alpha; \beta \cup \gamma \rangle_{\Gamma', d', E} = \sum_{d \rightarrow d'} \langle \alpha; \overline{\mathcal{M}}(\Upsilon)^* \beta \rangle_{\Gamma, d, E}$$

where γ is the dual class to $\overline{\mathcal{M}}(\Upsilon)$.

- (b) (Cutting an edge) *If $\Upsilon : \Gamma \rightarrow \Gamma'$ is of type cutting an edge then*

$$\langle \alpha; \beta \rangle_{\Gamma, d, E} = \sum_k \langle \alpha, \delta_k \cup \text{Eul}_{G \times \mathbb{C}^\times}(E), \delta^k; \overline{\mathcal{M}}(\Upsilon)^* \beta \rangle_{\Gamma', d, E}.$$

- (c) (Forgetting a tail) *If $\Upsilon : \Gamma \rightarrow \Gamma'$ is a morphism forgetting a tail, which corresponds to the last marking z_n then for $\alpha' \in H_G^2(X), \alpha \in H_G(X)^{n-1}$,*

$$\langle \alpha, \alpha'; \overline{\mathcal{M}}(\Upsilon)^* \beta \rangle_{\Gamma, d, E} = \langle d, \alpha' \rangle_{\Gamma', d', E} \langle \alpha; \beta \rangle_{\Gamma, d, E}$$

Proof. We discuss only the cutting-edge axiom; the rest are similar to those in the untwisted case. With notation as in (33), in particular with $p : \mathcal{C}' \rightarrow \mathcal{C}$ the normalization and x the section given by the node corresponding to the cut edge, the short exact sequence of sheaves

$$0 \rightarrow E \rightarrow p^* p_* E \rightarrow x^* x_* E \rightarrow 0$$

which gives rise to an exact triangle in the derived category of bounded complexes of coherent sheaves

$$R\pi_* \text{ev}^* E \rightarrow R\pi''_* p^* \text{ev}^* E \rightarrow x^* \text{ev}^* E \rightarrow R\pi_* \text{ev}^* E[1].$$

This implies that

$$\iota_{\Gamma', \Gamma}^* \text{Ind}(E) \cong \text{Ind}(E) \oplus E.$$

Taking Euler classes gives the result. \square

Define

$$QH_{G \times \mathbb{C}^\times}(X, \mathbb{Q}) = QH_G(X, \mathbb{Q}) \otimes \mathbb{Q}[\zeta, \zeta^{-1}]$$

and define twisted composition maps

$$\mu_E^{g,n} : QH_{G \times \mathbb{C}^\times}(X, \mathbb{Q})^n \times H(\overline{\mathcal{M}}_{g,n+1}, \mathbb{Q}) \rightarrow QH_{G \times \mathbb{C}^\times}(X, \mathbb{Q})$$

by

$$(\mu_E^{g,n}(\alpha_1, \dots, \alpha_n; \beta), \alpha_0) = \sum_{d \in H_2(X)} q^d \langle \alpha_0 \cup \text{Eul}_{G \times \mathbb{C}^\times}(E), \dots, \alpha_n; \beta \rangle_{g,d,E}.$$

Discussion of twisted composition maps can be found in e.g. Pandharipande [78].

Theorem 7.13. (Equivariant twisted Gromov-Witten invariants define a CohFT algebra) *Suppose that X is a smooth projective G -variety, and $E \rightarrow X$ be a G -equivariant vector bundle. The datum $(QH_{G \times \mathbb{C}^\times}(X, \mathbb{Q}), (\mu_E^{g,n})_{g,n \geq 0})$ form a CohFT algebra, denoted $QH_G(X, E)$.*

We leave it to the reader to check that inserting the Euler class $\epsilon(E)$ into the definition of the graph invariants (35) yields a twisted trace $\tau_{X,E} : QH_G(X, E) \rightarrow \Lambda_X$ generalizing Theorem 7.6.

7.4. Gauged Gromov-Witten invariants. In this section we define gauged Gromov-Witten invariants. As in Behrend [8], invariants are defined for any possibly disconnected combinatorial type, and the splitting axiom can be broken down into *cutting edges* and *collapsing edges* axiom. However, the definition for disconnected type requires an additional datum, of an assignment of each non-root component to a semi-infinite edge of a root component. This is because the non-root components have associated moduli spaces with *equivariant* virtual fundamental classes, while the root component has a non-equivariant virtual fundamental class. These combine to give a virtual fundamental class on the fiber product of the components over BG .

Let X be a smooth projective G -variety. For any rooted tree Γ and homology class $d \in H_2^G(X, \mathbb{Z})$ we denote by $\overline{\mathcal{M}}_{n,\Gamma}^G(C, X, d)$ the moduli stack of maps of combinatorial type Γ .

Definition 7.14. (Virtual fundamental classes for moduli stacks of gauged maps)

- (a) (Combinatorial type with a single vertex) We already remarked in Example 6.7 that if $\overline{\mathcal{M}}_n^G(C, X)$ is a Deligne-Mumford stack, then it has a perfect obstruction theory, given by the dual of the derived push-forward of the pull-back of the tangent complex $(Rp_* u^* T(X/G))^\vee$ where $p : \overline{\mathcal{C}}_n^G(X, d) \rightarrow \overline{\mathcal{M}}_n^G(X, d)$ is the universal curve, $u : \overline{\mathcal{C}}_n^G(X, d) \rightarrow X/G$ the universal stable gauged map, and $T(X/G)$ the tangent complex to X/G . Hence one obtains a virtual fundamental class of expected dimension

$$[\overline{\mathcal{M}}_n^G(C, X, d)] \in A(\overline{\mathcal{M}}_n^G(C, X, d)).$$

- (b) (Connected combinatorial type) More generally, given any connected rooted tree Γ we denote by $\overline{\mathcal{M}}_{n, \Gamma}^G(C, X, d)$ resp. $\overline{\mathcal{M}}_{n, \Gamma}^{G, \text{fr}}(C, X, d)$ the moduli stack of polystable gauged maps resp. with framings at the markings of combinatorial type Γ . Under the assumption that $\overline{\mathcal{M}}_n^G(C, X, d)$ is Deligne-Mumford, the action of G^n on $\overline{\mathcal{M}}_{n, \Gamma}^{G, \text{fr}}(C, X, d)$ is locally free. The same construction gives a virtual fundamental class

$$[\overline{\mathcal{M}}_{n, \Gamma}^G(C, X, d)] \in A(\overline{\mathcal{M}}_{n, \Gamma}^G(C, X, d)) \cong A^{G^n}(\overline{\mathcal{M}}_{n, \Gamma}^{G, \text{fr}}(C, X, d)).$$

- (c) (Disconnected combinatorial type) Suppose $\Gamma = \Gamma_0 \cup \dots \cup \Gamma_l$ with Γ_0 containing the root vertex is given the *additional data* of a map from the non-root components to the root edges: Suppose that $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \dots \cup \Gamma_l$ is a disconnected rooted $H_2^G(X, \mathbb{Z})$ -labelled graph such that Γ_j has semi-infinite edges I_j , and for each $j = 1, \dots, l$ is given a semi-infinite edge $e(j)$ of Γ_0 . We denote by

$$\overline{\mathcal{M}}_{n, \Gamma}^{G, f}(C, X, d) = (\overline{\mathcal{M}}_{n_0, \Gamma_0}^{G, \text{fr}}(C, X, d_0) \times \prod_{j=1}^l \overline{\mathcal{M}}_{0, n, \Gamma_j}(X, d_j)) / G^{n_0}$$

where the action of the i -th factor in G^{n_0} acts at the i -th framing on the principal component, and diagonally on the components corresponding to Γ_j with $e(j) = i$. We have virtual fundamental classes

$$[\overline{\mathcal{M}}_{n_0, \Gamma_0}^G(C, X, d_0)] \in A(\overline{\mathcal{M}}_{n_0, \Gamma_0}^G(C, X, d_0)) \cong A_{G^{n_0}}(\overline{\mathcal{M}}_{n_0, \Gamma_0}^{G, \text{fr}}(C, X, d_0))$$

$$[\overline{\mathcal{M}}_{0, n_j, \Gamma_j}(X, d_j)] \in A_G(\overline{\mathcal{M}}_{0, n_j, \Gamma_j}(X, d_j))$$

and so a virtual fundamental class

$$[\overline{\mathcal{M}}_{n, \Gamma}^G(C, X, d)] = \cup_{d=d_0+\dots+d_l} [\overline{\mathcal{M}}_{n_0, \Gamma_0}^{G, \text{fr}}(C, X, d_0)] \times \prod_{j=1}^l [\overline{\mathcal{M}}_{0, n, \Gamma_j}(X, d_j)]$$

in

$$A_{G^{n_0}}(\overline{\mathcal{M}}_{n_0, \Gamma_0}^{G, \text{fr}}(C, X, d) \times \prod_j \overline{\mathcal{M}}_{0, n_j}(X, d_j)) \cong A(\overline{\mathcal{M}}_{n, \Gamma}^{G, f}(C, X, d)).$$

These classes satisfy the following properties [8]:

- Proposition 7.15.** (a) (Constant maps) If $d = 0$ and $\text{genus}(C) = 0$ then $\overline{\mathcal{M}}_{\Gamma, n}^G(C, X, d) = (X//G) \times \overline{\mathcal{M}}_{\Gamma, n}(C)$ and $[\overline{\mathcal{M}}_{n, \Gamma}^G(C, X, d)] = [X//G \times \overline{\mathcal{M}}_{n, \Gamma}(C)]$.
- (b) (Products) If X_0, X_1 are G -varieties then $[\overline{\mathcal{M}}_{n, \Gamma}^{G \times G}(C, X_0 \times X_1, (d_0, d_1))] = [\overline{\mathcal{M}}_{n, \Gamma}^G(C, X_0, d_0)] \times [\overline{\mathcal{M}}_{n, \Gamma}^G(C, X_1, d_1)]$

- (c) (Cutting edges) If Γ' is obtained from Γ by cutting an edge then (with the obvious labelling of the additional component) $[\overline{\mathcal{M}}_{n,\Gamma}^G(C, X, d)] = \Delta^! [\overline{\mathcal{M}}_{n+2,\Gamma'}^G(C, X, d)]$.
- (d) (Collapsing edges) If $\Upsilon : \Gamma' \rightarrow \Gamma$ is a morphism collapsing an edge then $\overline{\mathcal{M}}(\Upsilon)^! [\overline{\mathcal{M}}_{n,\Gamma}^G(C, X, d)]$ is the push-forward of $\sum_{d' \mapsto d} [\overline{\mathcal{M}}_{n,\Gamma'}^G(C, X, d')]$ under

$$\overline{\mathcal{F}}(\Upsilon, X) : \overline{\mathcal{M}}_{n,\Gamma'}^G(C, X, d') \rightarrow \overline{\mathcal{M}}_{n,\Gamma'}(C) \times_{\overline{\mathcal{M}}_{n,\Gamma}(C)} \overline{\mathcal{M}}_{n,\Gamma}^G(C, X, d).$$

- (e) (Forgetting tails) If $\Upsilon : \Gamma \rightarrow \Gamma'$ is a morphism forgetting a tail then

$$\overline{\mathcal{M}}(\Upsilon)^! [\overline{\mathcal{M}}_{n,\Gamma'}^G(C, X, d)] = [\overline{\mathcal{M}}_{n,\Gamma}^G(C, X, d)].$$

Proof. The items (a), (b) and (e) are similar to the ordinary Gromov-Witten case considered in Behrend [8] and left to the reader. For a morphism Υ cutting an edge for gauged maps, recall from Proposition 5.21 that $\overline{\mathcal{M}}_{n,\Gamma}^G(C, X)$ may be identified with the fiber product $\overline{\mathcal{M}}_{n,\Gamma'}^G(C, X) \times_{(X/G)^2} (X/G)$ over the diagonal $\Delta : (X/G) \rightarrow (X/G)^2$. We denote by

$$\overline{\mathcal{M}}(\Upsilon, X) : \overline{\mathcal{M}}_{n,\Gamma}^G(C, X) \rightarrow \overline{\mathcal{M}}_{n,\Gamma'}^G(C, X)$$

the resulting morphism. We check that the obstruction theories $E_{\Gamma'}$ and E_{Γ} are compatible over Δ . Let \mathcal{C} denote the universal curve over $\overline{\mathcal{M}}_{n,\Gamma}^G(C, X)$, similarly for \mathcal{C}' and Γ' . Let $\mathcal{C}'' = \overline{\mathcal{M}}(\Upsilon, X)^* \mathcal{C}'$ be the curve over $\overline{\mathcal{M}}_{n,\Gamma'}^G(C, X)$ obtained by normalizing at the node corresponding to the edge, with $f : \mathcal{C}'' \rightarrow \mathcal{C}$ the projection and $e'' : \mathcal{C}'' \rightarrow X/G$, $e : \mathcal{C} \rightarrow X/G$ the universal maps. So \mathcal{C} is obtained from \mathcal{C}'' by identifying the two sections x_1, x_2 of \mathcal{C}' , and is equipped with a section x induced from x_1, x_2 . The short exact sequence of complexes of coherent sheaves

$$0 \rightarrow e^* T_{X/G} \rightarrow f_* f^* e^* T_{X/G} \rightarrow x_* x^* e^* T_{X/G} \rightarrow 0$$

(viewing $T_{X/G}$ as a two-term complex) induces an exact triangle in the derived category

$$Rp_* e^* T_{X/G} \rightarrow Rp_*'' f^* e^* T_{X/G} \rightarrow x^* e^* T_{X/G} \rightarrow Rp_* e^* T_{X/G}[1].$$

We have relative obstruction theories with complexes

$$E_{\Gamma} := (Rp_* e^* T_{X/G})^{\vee}, \quad \overline{\mathcal{M}}(\Upsilon, X)^* E_{\Gamma'} = (Rp_*'' e''^* T_{X/G})^{\vee} = (Rp_*'' f^* e^* T_{X/G})^{\vee}.$$

Note that $(x^* e^* T_{X/G})^{\vee} = \psi^* L_{\Delta}$, where $\Delta : (X/G) \rightarrow (X/G)^2$ is the diagonal and ψ is evaluation at the node. We have an exact triangle

$$e^* L_{\Delta}[-1] \rightarrow \overline{\mathcal{M}}(\Upsilon, X)^* E_{\Gamma'} \rightarrow E_{\Gamma} \rightarrow e^* L_{\Delta}.$$

This gives rise to a morphism of exact triangles

$$\begin{array}{ccccccc} \overline{\mathcal{M}}(\Upsilon, X)^* E_{\Gamma'} & \longrightarrow & E_{\Gamma} & \longrightarrow & e^* L_{\Delta} & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ \overline{\mathcal{M}}(\Upsilon, X)^* L_{\overline{\mathcal{M}}_{n,\Gamma'}^G(C, X)/\overline{\mathcal{M}}_{n,\Gamma'}(C)} & \longrightarrow & L_{\overline{\mathcal{M}}_{n,\Gamma}^G(C, X)/\overline{\mathcal{M}}_{n,\Gamma}(C)} & \longrightarrow & L_{\overline{\mathcal{M}}(\Upsilon, X)} & \longrightarrow & \cdots \end{array}$$

By compatibility, the virtual fundamental classes are related by $[\overline{\mathcal{M}}_{n,\Gamma}^G(C, X)] = \Delta^! [\overline{\mathcal{M}}_{n,\Gamma'}^G(C, X)]$.

For collapsing an edge for gauged maps, let $\Upsilon : \Gamma' \rightarrow \Gamma$ be a morphism of rooted graphs given by *collapsing an edge*. Associated to Υ are morphisms of Artin resp. Deligne-Mumford stacks

$$\overline{\mathfrak{M}}(\Upsilon) : \overline{\mathfrak{M}}_{n,\Gamma'}(C) \rightarrow \overline{\mathfrak{M}}_{n,\Gamma}(C), \quad \overline{\mathcal{M}}(\Upsilon) : \overline{\mathcal{M}}_{n,\Gamma'}(C) \rightarrow \overline{\mathcal{M}}_{n,\Gamma}(C).$$

The first is a regular local immersion, and so defines a class in the bivariant Chow group $[\overline{\mathfrak{M}}(\Upsilon)] \in A^\vee(\overline{\mathfrak{M}}_{n,\Gamma'}(C) \rightarrow \overline{\mathfrak{M}}_{n,\Gamma}(C))$. As in Behrend [8], the relative obstruction theories for $\overline{\mathcal{M}}_{n,\Gamma}^G(C, X), \overline{\mathcal{M}}_{n,\Gamma'}^G(C, X)$ are related by pull-back. Indeed consider the diagram analogous to [8, p. 15] in Figure 16. Because all the squares are Cartesian, it follows as in [8] that

$$\begin{array}{ccccc} \sqcup_{d' \mapsto d} \overline{\mathcal{M}}_{n,\Gamma'}^G(C, X, d') & \longrightarrow & \overline{\mathcal{M}}_{n,\Gamma'}(C) \times_{\overline{\mathcal{M}}_{n,\Gamma}(C)} \overline{\mathcal{M}}_{n,\Gamma}^G(C, X, d) & \longrightarrow & \overline{\mathcal{M}}_{n,\Gamma}^G(C, X, d) \\ \downarrow & & \downarrow & & \downarrow \\ \overline{\mathfrak{M}}_{n,\Gamma'}(C) & \longrightarrow & \overline{\mathcal{M}}_{n,\Gamma'}(C) \times_{\overline{\mathcal{M}}_{n,\Gamma}(C)} \overline{\mathfrak{M}}_{n,\Gamma}(C) & \longrightarrow & \overline{\mathfrak{M}}_{n,\Gamma}(C) \\ & \searrow & \downarrow & & \downarrow \\ & & \overline{\mathcal{M}}_{n,\Gamma'}(C) & \longrightarrow & \overline{\mathcal{M}}_{n,\Gamma}(C) \end{array}$$

FIGURE 16. Diagram for collapsing an edge

$$\overline{\mathfrak{M}}(\Upsilon)^! [\overline{\mathcal{M}}_{n,\Gamma}^G(C, X, d)] = \overline{\mathcal{F}}(\Upsilon, X)_* \sum_{d' \mapsto d} [\overline{\mathcal{M}}_{n,\Gamma'}^G(C, X, d')]$$

as claimed. \square

We now pass to homology/cohomology. (Of course one could consider the quantum Chow ring etc.) Pairing with the virtual fundamental class gives a map

$$\int_{[\overline{\mathcal{M}}_n^G(C, X)]} : H(\overline{\mathcal{M}}_n^G(C, X), \mathbb{Q}) \rightarrow \mathbb{Q}.$$

Evaluation at the marked points gives a morphism

$$\text{ev} : \overline{\mathcal{M}}_n^G(C, X) \rightarrow (X/G)^n, \quad (P, \hat{C}, u, z_1, \dots, z_n) \mapsto (z_j^* P, u \circ z_j)_{j=1}^n.$$

Forgetting the bundle and curve and collapsing any unstable components defines a forgetful morphism from Proposition 5.19 $f : \overline{\mathcal{M}}_n^G(C, X) \rightarrow \overline{\mathcal{M}}_n(C)$.

Definition 7.16. (Gauged Gromov-Witten invariants of a given combinatorial type)

- (a) (Invariants for a tree with a single vertex) The *gauged Gromov-Witten invariants* associated to X are the maps

$$H_G(X, \mathbb{Q})^n \times H(\overline{\mathcal{M}}_n(C), \mathbb{Q}) \rightarrow \mathbb{Q}, \quad (\alpha, \beta) \mapsto \langle \alpha, \beta \rangle_d$$

$$\langle \alpha; \beta \rangle_d := \int_{[\overline{\mathcal{M}}_n^G(C, X, d)]} \text{ev}^* \alpha \cup f^* \beta.$$

- (b) (Invariants for a connected tree) Invariants for a *connected* rooted $H_2^G(X, \mathbb{Z})$ -labelled tree Γ and G -equivariant vector bundle $E \rightarrow X$ are defined as follows: let $\langle \alpha, \beta \rangle_{E, \Gamma, d} \in \mathbb{Q}$ defined by integration of $\text{ev}^* \alpha \cup f^* \beta \cup \epsilon(E)$ over the moduli stack $\overline{\mathcal{M}}_{n, \Gamma}^G(C, X)$ of stable gauged maps of combinatorial type Γ .
- (c) (Invariants for forests) Invariants for possibly $H_2^G(X, \mathbb{Z})$ -labeled rooted forests are defined as follows, given the *additional data* of a map from the non-root components to the root edges: Suppose that $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \dots \cup \Gamma_l$ is a rooted $H_2^G(X, \mathbb{Z})$ -labelled forest such that each tree Γ_j has semi-infinite edges I_j , and for each $j = 1, \dots, l$ is given a semi-infinite edge $e(j)$ of Γ_0 . We define gauged Gromov-Witten invariants for Γ by fiber integration over the map $\overline{\mathcal{M}}_{n, \Gamma}^G(C, X) \rightarrow \overline{\mathcal{M}}_{n_0, \Gamma_0}^G(C, X)$ whose fibers are moduli stack of stable maps of type Γ_j for $j > 0$: set

$$\langle \alpha; \beta \rangle_{\Gamma, d} := \langle (\alpha'_i)_{i \in I_0}; \beta \rangle_{\Gamma_0, d_0}$$

where for each semi-infinite edge i of Γ_0

$$\alpha'_i = \left(\prod_{e \in I_j} \langle (\alpha_e)_{e \in I_j}, \beta_e \rangle_{\Gamma_j, d_j} \right) \alpha_i$$

using the $H(BG)$ -module structure on $H_G(X)$, where $\beta_j \in H(\overline{\mathcal{M}}_{n_j, \Gamma_j}(C))$ is the component of β in the decomposition $\overline{\mathcal{M}}_{n, \Gamma}(C) = \prod_j \overline{\mathcal{M}}_{n_j, \Gamma_j}(C)$.

Remark 7.17. It is not possible to define invariants for forests (as opposed to trees) as purely a product over the tree components, since the non-root components resp. root component defines invariants with values in $H(BG) \otimes \Lambda_X^G$ resp. Λ_X^G .

These invariants (with or without cohomological twisting) satisfy axioms for morphisms of rooted trees:

Proposition 7.18. (a) (Cutting edges) *If Γ' is obtained from Γ by cutting an edge then*

$$\langle \alpha; \beta \rangle_{\Gamma, d} = \sum_{i=1}^{\dim(H(X))} \langle \alpha, \delta_i, \delta^i; \overline{\mathcal{M}}(\Upsilon)^* \beta \rangle_{\Gamma', d}$$

where δ_i, δ^i are dual bases for $H_G(X)$ over $H(BG)$;

- (b) (Collapsing edges) *If $\Upsilon : \Gamma \rightarrow \Gamma'$ is a morphism collapsing an edge then*

$$\langle \alpha; \beta \cup \gamma \rangle_{\Gamma, d} = \sum_{d' \mapsto d} \langle \alpha; \overline{\mathcal{M}}(\Upsilon)^* \beta \rangle_{\Gamma', d'}$$

where $\gamma \in H^2(\overline{\mathcal{M}}_{n, \Gamma}(C))$ is the dual class for $\overline{\mathcal{M}}(\Upsilon) : \overline{\mathcal{M}}_{n, \Gamma}(C) \rightarrow \overline{\mathcal{M}}_{n, \Gamma'}(C)$.

- (c) (Forgetting tails) If $\Upsilon : \Gamma \rightarrow \Gamma'$ is a morphism forgetting a tail then for $\alpha' \in H_G^2(X)$, $\alpha \in H_G(X)^{n-1}$,

$$\langle \alpha, \alpha'; \overline{\mathcal{M}}(\Upsilon)^* \beta \rangle_{\Gamma, d} = (d, \alpha') \langle \alpha; \beta \rangle_{\Gamma', d'}$$

Proof. By Proposition 7.15. \square

Definition 7.19. Denote by Λ_X^G the *equivariant Novikov field* for X , the set of all maps $a : H_2^G(X) := H_2^G(X, \mathbb{Z}) / \text{torsion} \rightarrow \mathbb{Q}$ such that for every constant c , the set of classes

$$\{d \in H_2^G(X), \langle [\omega_{X,G}], d \rangle \leq c\}$$

on which a is non-vanishing is finite. Addition is defined in the usual way and multiplication is convolution.

From now on, we denote by $QH_G(X) = H_G(X) \otimes \Lambda_X^G$ the quantum cohomology over the Novikov field Λ_X^G . Summing over equivariant homology classes gives a map

$$\tau_{X,n}^G : QH_G(X)^n \times H(\overline{\mathcal{M}}_n(C)) \rightarrow \Lambda_X^G, \quad \sum_{d \in H_2^G(X, \mathbb{Z})} q^d \langle \alpha, \beta \rangle_d.$$

By Proposition 7.18,

Theorem 7.20. If $\overline{\mathcal{M}}_n^G(C, X)$ is a Deligne-Mumford stack (that is, if stable=semistable) then the maps $(\tau_{X,n}^G)_{n \geq 0}$ form a trace on the CohFT algebra $QH_G(X)$.

Twisted gauged Gromov-Witten invariants are defined as follows.

Definition 7.21. (Twisting class and twisted gauged invariants) Let $E \rightarrow X$ be a G -equivariant complex vector bundle, inducing a vector bundle on X/G . Pull-back under the evaluation map $e : \overline{\mathcal{C}}_n^G(C, X) \rightarrow X/G$ gives rise to a vector bundle $e^*E \rightarrow \overline{\mathcal{C}}_n^G(C, X)$, which we can push down to an *index* complex

$$\text{Ind}(E) := Rp_* e^* E$$

in the derived category of bounded complexes of coherent sheaves on $\overline{\mathcal{M}}_n^G(C, X)$. As in [17, Appendix], $\text{Ind}(E)$ admits a resolution by vector bundles and we may define the \mathbb{C}^\times -equivariant Euler class

$$\epsilon(E) := \text{Eul}_{\mathbb{C}^\times}(\text{Ind}(E)) \in H_G(\overline{\mathcal{M}}_n^G(C, X)) \otimes \mathbb{Q}[\zeta, \zeta^{-1}]$$

where ζ is the parameter for the action of \mathbb{C}^\times by scalar multiplication in the fibers. The *twisted gauged Gromov-Witten invariants* associated to $E \rightarrow X, C$ are the maps

$$(40) \quad H_G(X)^n \times H(\overline{\mathcal{M}}_n(C)) \rightarrow \mathbb{Q}[\zeta, \zeta^{-1}],$$

$$(\alpha, \beta) \mapsto \langle \alpha, \beta \rangle_{\Gamma, E, d} = \int_{[\overline{\mathcal{M}}_n^G(C, X, d)]} \text{ev}^* \alpha \cup f^* \beta \cup \epsilon(E).$$

Example 7.22. Recall that in the case that X is a vector space, G is a torus, and E is a vector bundle corresponding to a Calabi-Yau hypersurface in $X//G$, $\overline{\mathcal{M}}^G(C, X)$ is the toric variety $X(d)$ of (30). Integrals over toric varieties may be computed via residues, as in for example Szenes-Vergne [87]. Some sample computations are computed in Morrison-Plesser [64, Section 4], who made contact with the Gelfand-Kapranov-Zelevinsky theory of hypergeometric functions. We return to this case in Example 9.21.

8. QUANTUM KIRWAN MORPHISM AND THE ADIABATIC LIMIT THEOREM

In this section we explain how to “quantize” the classical Kirwan morphism $H_G(X) \rightarrow H(X//G)$ in order to obtain a morphism of CohFT algebras from $QH_G(X)$ to $QH(X//G)$. The existence of such a morphism was noted under “sufficiently positive” conditions on the first Chern class in Gaio-Salamon [29]. The quantum Kirwan morphism relates small quantum cohomologies under suitable positivity assumptions. We also give a partial computation of the quantum Kirwan map in the toric case.

8.1. Affine gauged Gromov-Witten invariants. We first define gauged affine Gromov-Witten invariants by integrating pull-back and universal classes over the moduli stack of affine gauged maps. As in Behrend [8], we separate the splitting axiom into a *cutting edges* and *collapsing edges* axiom. The main difference with Behrend [8] is that one cannot cut an arbitrary edge and still have a colored tree if the edge separates some of the colored vertices from the root edge and not others, so there is a new *cutting edges with relations* axiom which cuts several edges at once. There is also a difference in the *collapsing edges* axiom: because the source moduli space $\overline{\mathcal{M}}_{n,1}(\mathbb{A})$ is not smooth, not every boundary divisor is Cartier and so there is a new *collapsing edges with relations* axiom which holds for combinations of boundary divisors that are Cartier.

In this section X is a smooth polarized quasiprojective variety such that the git quotient $X//G$ is a (necessarily smooth) Deligne-Mumford stack. To define virtual fundamental classes, note that $\overline{\mathcal{M}}_{n,1}^G(\mathbb{A}, X)$ admits a forgetful morphism to $\overline{\mathfrak{M}}_{n,1}^{\text{tw}}(\mathbb{A})$ (where the superscript *tw* indicates that we orbifold structures at the nodes with infinite scaling, in the case that $X//G$ is only locally free) and to $\overline{\mathcal{M}}_{n,1}(\mathbb{A})$, the latter collapsing components that become unstable after forgetting the morphism to X/G . $\overline{\mathcal{M}}_{n,1}^G(\mathbb{A}, X)$ has a canonical perfect relative obstruction theory over the stack of not-necessarily-stable scaled marked curves $\overline{\mathfrak{M}}_{n,1}^{\text{st}}(\mathbb{A})$, whose complex is dual to the push-forward of $u^*T(X/G)$ over the universal curve over $\overline{\mathcal{M}}_{n,1}^G(\mathbb{A}, X)$ from Example 6.7.

Definition 8.1. (Virtual fundamental classes for affine gauged maps)

- (a) (Virtual fundamental class for a colored tree with a single vertex) The construction in [9, Chapter 7] gives a virtual fundamental class $[\overline{\mathcal{M}}_{n,1}^G(\mathbb{A}, X, d)] \in A(\overline{\mathcal{M}}_{n,1}^G(\mathbb{A}, X, d))$.
- (b) (Virtual fundamental class for a connected colored tree) More generally, for any combinatorial type of colored tree Γ we have virtual fundamental classes

$$[\overline{\mathcal{M}}_{n,1,\Gamma}^G(\mathbb{A}, X, d)] \in A(\overline{\mathcal{M}}_{n,1,\Gamma}^G(\mathbb{A}, X, d)).$$

- (c) (Virtual fundamental class for a disconnected colored forest) Suppose that $\Gamma = \Gamma_0 \cup \dots \cup \Gamma_l$ with Γ_0 a possibly disconnected union of components each with at least one colored vertex, and $\Gamma_1, \dots, \Gamma_l$ connected components with $\text{Vert}(\Gamma_j) \subset \text{Vert}^0(\Gamma)$. Suppose that for each component Γ_j we are given an non-root edge $e(j)$ of Γ_0 . We denote by

$$\overline{\mathcal{M}}_{n,\Gamma}^{G,f}(\mathbb{A}, X, d) := \cup_{d=d_0+\dots+d_l} \overline{\mathcal{M}}_{n_0,\Gamma_0}^{G,\text{fr}}(\mathbb{A}, X, d_0) \times_{G^{n_0}} \prod_{j=1}^l \overline{\mathcal{M}}_{0,n,\Gamma_j}(X, d_j)$$

the fiber product determined by the mapping e above. We have virtual fundamental classes

$$[\overline{\mathcal{M}}_{n_0, \Gamma_0}^G(\mathbb{A}, X, d_0)] \in A(\overline{\mathcal{M}}_{n_0, \Gamma_0}^G(\mathbb{A}, X, d)) \cong A_{G^{n_0}}(\overline{\mathcal{M}}_{n_0, \Gamma_0}^{G, \text{fr}}(\mathbb{A}, X, d))$$

given the product of virtual fundamental classes of the components and equivariant virtual fundamental classes

$$[\overline{\mathcal{M}}_{0, n_j, \Gamma_j}(X, d_j)] \in A_G(\overline{\mathcal{M}}_{0, n_j, \Gamma_j}(X, d_j)).$$

These give a virtual fundamental class

$$[\overline{\mathcal{M}}_{n, \Gamma}^G(\mathbb{A}, X, d)] = \cup_{d=d_0+\dots+d_l} [\overline{\mathcal{M}}_{n_0, \Gamma_0}^{G, \text{fr}}(\mathbb{A}, X, d_0)] \times \prod_{j=1}^l [\overline{\mathcal{M}}_{0, n_j, \Gamma_j}(X, d_j)]$$

in

$$A_{G^{n_0}}(\overline{\mathcal{M}}_{n_0, \Gamma_0}^{G, \text{fr}}(\mathbb{A}, X, d) \times \prod_j \overline{\mathcal{M}}_{0, n_j, \Gamma_j}(X, d_j)) \cong A(\overline{\mathcal{M}}_{n, \Gamma}^{G, f}(\mathbb{A}, X, d)).$$

Note that it is not possible to define the virtual fundamental classes without the additional labelling, since the virtual fundamental classes for the components Γ_j are equivariant while that for Γ_0 is not.

These classes satisfy the following properties:

Proposition 8.2. (a) (Collapsing edges) *If Γ' is obtained from Γ by collapsing an edge and $\Upsilon : \Gamma \rightarrow \Gamma'$ is the corresponding morphism of colored trees then*

$$\overline{\mathfrak{M}}(\Upsilon)^! [\overline{\mathcal{M}}_{n, 1, \Gamma'}^G(\mathbb{A}, X, d')] = \overline{\mathcal{F}}(\Upsilon, X)_* \sum_{d \rightarrow d'} [\overline{\mathcal{M}}_{n, 1, \Gamma}^G(\mathbb{A}, X, d)]$$

where

$$\overline{\mathcal{F}}(\Upsilon, X) : \overline{\mathcal{M}}_{n, 1, \Gamma'}^G(\mathbb{A}, X, d') \rightarrow \overline{\mathcal{M}}_{n, 1, \Gamma'}(\mathbb{A}) \times_{\overline{\mathcal{M}}_{n, 1, \Gamma}(\mathbb{A})} \overline{\mathcal{M}}_{n, 1, \Gamma}^G(\mathbb{A}, X, d)$$

is the identification with the fiber product;

- (b) (Collapsing edges with relations) *If $\Gamma_0, \dots, \Gamma_r$ are obtained from Γ by collapsing edges with relations and $\Upsilon : \Gamma_0 \sqcup \dots \sqcup \Gamma_r \rightarrow \Gamma$ is the corresponding morphism of colored trees so that $\cup_{i=1}^r \overline{\mathfrak{M}}_{n, 1, \Gamma_i}^{\text{tw}}(\mathbb{A}) \rightarrow \overline{\mathfrak{M}}_{n, 1, \Gamma'}^{\text{tw}}(\mathbb{A})$ is a regular local immersion (that is, is a Cartier divisor) then*

$$\overline{\mathfrak{M}}(\Upsilon)^! [\overline{\mathcal{M}}_{n, 1, \Gamma}^G(\mathbb{A}, X, d)] = \overline{\mathcal{F}}(\Upsilon, X)_* \sum_{d \rightarrow d', i=1, \dots, r} [\overline{\mathcal{M}}_{n, 1, \Gamma_i}^G(\mathbb{A}, X, d)].$$

- (c) (Cutting edges or edges with relations) *If $\Upsilon : \Gamma \rightarrow \Gamma'$ is a morphism of trees of type cutting an edge or edges with relations then*

$$\mathcal{G}(\Upsilon, X)_* [\overline{\mathcal{M}}_{n, \Gamma'}^G(\mathbb{A}, X, d')] = \Delta^! [\overline{\mathcal{M}}_{n, \Gamma}^{G, f}(\mathbb{A}, X, d)]$$

where $\Delta : I_{X/G}^m \rightarrow I_{X/G}^{2m}$ is the diagonal and $\mathcal{G}(\Upsilon, X)$ is the gluing morphism in (31).

- (d) (Forgetting tails) *If $\Upsilon : \Gamma \rightarrow \Gamma'$ is a morphism forgetting a tail then*

$$\overline{\mathcal{M}}(\Upsilon)^! [\overline{\mathcal{M}}_{n, \Gamma'}^G(\mathbb{A}, X, d)] = [\overline{\mathcal{M}}_{n, \Gamma}^G(\mathbb{A}, X, d)].$$

Proof. Cutting an edge is similar to the case of gauged maps from projective curves covered in Proposition 7.15 and omitted. For collapsing an edge, Let $\Upsilon : \Gamma' \rightarrow \Gamma$ be a morphism of edge-rooted colored trees given by *collapsing an edge* connected vertices of the same color. Associated to Υ are morphisms of Artin resp. Deligne-Mumford stacks

$$\overline{\mathfrak{M}}(\Upsilon) : \overline{\mathfrak{M}}_{n,1,\Gamma'}^{\text{tw}}(\mathbb{A}) \rightarrow \overline{\mathfrak{M}}_{n,1,\Gamma}^{\text{tw}}(\mathbb{A}), \quad \overline{\mathcal{M}}(\Upsilon) : \overline{\mathcal{M}}_{n,1,\Gamma'}(\mathbb{A}) \rightarrow \overline{\mathcal{M}}_{n,1,\Gamma}(\mathbb{A}).$$

The first is a regular local immersion, and so defines a class in the bivariant Chow group

$$[\overline{\mathfrak{M}}(\Upsilon)] \in A^\vee(\overline{\mathfrak{M}}_{n,1,\Gamma'}^{\text{tw}}(\mathbb{A}) \rightarrow \overline{\mathfrak{M}}_{n,1,\Gamma}^{\text{tw}}(\mathbb{A})).$$

As in Behrend [8], the relative obstruction theories for $\overline{\mathcal{M}}_{n,1,\Gamma}^G(\mathbb{A}, X), \overline{\mathcal{M}}_{n,1,\Gamma'}^G(\mathbb{A}, X)$ are related by pull-back:

$$\overline{\mathfrak{M}}(\Upsilon)^! [\overline{\mathcal{M}}_{n,1,\Gamma'}^G(\mathbb{A}, X, d')] = \overline{\mathcal{F}}(\Upsilon, X)_* \sum_{d \rightarrow d'} [\overline{\mathcal{M}}_{n,1,\Gamma}^G(\mathbb{A}, X, d)].$$

Indeed consider the diagram analogous to [8, p. 15]

$$\begin{array}{ccccc} \sqcup_{d' \rightarrow d} \overline{\mathcal{M}}_{n,1,\Gamma'}^G(\mathbb{A}, X, d') & \longrightarrow & \overline{\mathcal{M}}_{n,1,\Gamma'}(\mathbb{A}) \times_{\overline{\mathcal{M}}_{n,1,\Gamma}(\mathbb{A})} \overline{\mathcal{M}}_{n,1,\Gamma}^G(\mathbb{A}, X, d) & \longrightarrow & \overline{\mathcal{M}}_{n,1,\Gamma}^G(\mathbb{A}, X, d) \\ \downarrow & & \downarrow & & \downarrow \\ \overline{\mathfrak{M}}_{n,1,\Gamma'}^{\text{tw}}(\mathbb{A}) & \longrightarrow & \overline{\mathcal{M}}_{n,1,\Gamma'}^{\text{tw}}(\mathbb{A}) \times_{\overline{\mathcal{M}}_{n,1,\Gamma}(\mathbb{A})} \overline{\mathfrak{M}}_{n,1,\Gamma}^{\text{tw}}(\mathbb{A}) & \longrightarrow & \overline{\mathfrak{M}}_{n,1,\Gamma}^{\text{tw}}(\mathbb{A}) \\ & \searrow & \downarrow & & \downarrow \\ & & \overline{\mathcal{M}}_{n,1,\Gamma'}^{\text{tw}}(\mathbb{A}) & \longrightarrow & \overline{\mathcal{M}}_{n,1,\Gamma}^{\text{tw}}(\mathbb{A}) \end{array}$$

Because all the squares are Cartesian, it follows as in [8] that

$$(41) \quad \overline{\mathfrak{M}}(\Upsilon)^! [\overline{\mathcal{M}}_{n,1,\Gamma'}^G(\mathbb{A}, X, d')] = \overline{\mathcal{F}}(\Upsilon, X)_* \sum_{d \rightarrow d'} [\overline{\mathcal{M}}_{n,1,\Gamma}^G(\mathbb{A}, X, d)].$$

For collapsing several edges, let $\Gamma_0, \dots, \Gamma_r$ be colored trees obtained from Γ by collapsing edges by morphisms $\Upsilon_1, \dots, \Upsilon_r$ so that $\cup_{i=1}^r \overline{\mathfrak{M}}_{n,1,\Gamma_i}^{\text{tw}}(\mathbb{A}) \rightarrow \overline{\mathfrak{M}}_{n,1,\Gamma}^{\text{tw}}(\mathbb{A})$ is a regular local immersion (that is, is a Cartier divisor). Then

$$\overline{\mathfrak{M}}(\Upsilon)^! [\overline{\mathcal{M}}_{n,1,\Gamma'}^G(\mathbb{A}, X, d')] = \overline{\mathcal{F}}(\Upsilon, X)_* \sum_{d \rightarrow d'} [\overline{\mathcal{M}}_{n,1,\Gamma}^G(\mathbb{A}, X, d)]$$

by the same argument as in the previous example. The last item is left to the reader. \square

To define invariants, note that evaluation at the marked points defines a map

$$\text{ev} \times \text{ev}_\infty : \overline{\mathcal{M}}_{n,1}^G(\mathbb{A}, X) \rightarrow (X/G)^n \times \overline{I}_{X//G}.$$

By integration over the moduli stacks of affine gauged maps we obtain affine gauged Gromov-Witten invariants defining the quantum Kirwan morphism of CohFT algebras from $QH_G(X)$ to $QH(X//G)$.

Definition 8.3. (Affine gauged Gromov-Witten invariants)

- (a) (Invariants for a connected colored tree) The *affine gauged Gromov-Witten invariants* for a connected colored tree Γ are the maps

$$(42) \quad H_G(X)^n \times H(X//G) \times H(\overline{\mathcal{M}}_{n,1}(\mathbb{A})) \rightarrow \mathbb{Q},$$

$$(\alpha, \alpha_\infty, \beta) \mapsto \langle \alpha; \alpha_\infty; \beta \rangle_{\Gamma, d} := \int_{[\overline{\mathcal{M}}_{n,1,\Gamma}^G(\mathbb{A}, X, d)]} \text{ev}^* \alpha \cup f^* \beta \cup \text{ev}_\infty^* \alpha_\infty.$$

- (b) (Invariants for a colored forest) Invariants for possibly disconnected $H_2^G(X, \mathbb{Z})$ -labeled colored forests are defined as follows, given the *additional data* of a map from the non-root components to the root edges: Suppose that $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \dots \cup \Gamma_l$ is a disconnected colored $H_2^G(X, \mathbb{Z})$ -labelled tree such that each component of Γ_0 has at least one vertex in $\text{Vert}^0(\Gamma)$ or $\text{Vert}^1(\Gamma)$, for $j > 1$ the tree Γ_j has semi-infinite edges labelled I_j , and for each $j = 1, \dots, l$ is given a semi-infinite edge $e(j)$ of Γ_0 . Let $\text{Edge}(\Gamma) = \text{Edge}^0(\Gamma) \cup E^\infty(\Gamma)$ denote the partition corresponding to nodes mapping to X/G or $I_{X//G}$, that is, edges connecting $\text{Vert}^0(\Gamma)$ with $\text{Vert}^0(\Gamma) \cup \text{Vert}^1(\Gamma)$ or edges connecting $\text{Vert}^1(\Gamma) \cup \text{Vert}^\infty(\Gamma)$ with $\text{Vert}^\infty(\Gamma)$ as in Remark 2.29. We suppose that we have a labelling of the semi-infinite edges by classes $\alpha_e \in H_G(X), e \in \text{Edge}^0(\Gamma)$ and $\alpha_e \in H(I_{X//G}), e \in \text{Edge}^\infty(\Gamma)$. We define gauged Gromov-Witten invariants for Γ by fiber integration over the map $\overline{\mathcal{M}}_{n,\Gamma}^G(\mathbb{A}, X) \rightarrow \overline{\mathcal{M}}_{n_0,\Gamma_0}^G(\mathbb{A}, X)$ whose fibers are moduli stack of stable maps of type Γ_j for $j > 0$: set

$$\langle \alpha; \beta \rangle_{\Gamma, d} := \langle (\alpha'_j)_{j \in I_0}; \beta \rangle_{\Gamma_0, d_0}$$

where for each semi-infinite edge i of Γ_0 connecting to a vertex in $\text{Vert}^0(\Gamma_0)$ or $\text{Vert}^1(\Gamma_0)$,

$$\alpha'_i = \left(\prod_{i=e(j)} \langle (\alpha_e)_{e \in I_j}, \beta_e \rangle_{\Gamma_j, d_j} \right) \alpha_i$$

using the $H(BG)$ -module structure on $H_G(X)$, where $\beta_j \in H(\overline{\mathcal{M}}_{n_j, \Gamma_j}(\mathbb{A}))$ is the Künneth component of β in the decomposition $\overline{\mathcal{M}}_{n, \Gamma}(\mathbb{A}) = \prod_j \overline{\mathcal{M}}_{n_j, \Gamma_j}(\mathbb{A})$.

- (c) (Twisted affine Gromov-Witten invariants) Twisted invariants $\langle \alpha; \beta \rangle_{\Gamma, d, E}$ associated to G -equivariant vector bundles $E \rightarrow X$ are defined by inserting Euler classes of indices $\epsilon(E)$ into the integrands.

The properties of the affine Gromov-Witten invariants are similar to those for the projective case:

Proposition 8.4. (a) (Collapsing an edge) *If Γ' is obtained from Γ by collapsing an edge then for any labelling d' of Γ' , and $\overline{\mathcal{M}}(\Upsilon) : \overline{\mathcal{M}}_{n,1,\Gamma}^{\text{tw}}(\mathbb{A}) \rightarrow \overline{\mathcal{M}}_{n,1,\Gamma'}^{\text{tw}}(\mathbb{A})$ has dual class γ then*

$$\langle \alpha; \beta \cup \gamma \rangle_{\Gamma', d', E} = \sum_{d \rightarrow d'} \langle \alpha; \overline{\mathcal{M}}(\Upsilon)^* \beta \rangle_{\Gamma', d, E}$$

(b) (Collapsing edges with relations) *More generally, if $\Gamma_0, \dots, \Gamma_r$ are each obtained from Γ by collapsing edges with relations and $\Upsilon : \Gamma_0 \sqcup \dots \sqcup \Gamma_r \rightarrow \Gamma$ is the corresponding morphism of colored trees so that*

$$(43) \quad \cup_{i=1}^r \overline{\mathcal{M}}_{n,1,\Gamma_i}^{\text{tw}}(\mathbb{A}) \rightarrow \overline{\mathcal{M}}_{n,1,\Gamma'}^{\text{tw}}(\mathbb{A})$$

is a regular local immersion (that is, is a Cartier divisor) with dual class γ then

$$\langle \alpha; \beta \cup \gamma \rangle_{\Gamma', d', E} = \sum_{d \rightarrow d', i=1, \dots, r} \langle \alpha; \iota_{\Gamma_i, \Gamma}^* \beta \rangle_{\Gamma_i, d, E}$$

where γ is the dual class to $\overline{\mathcal{M}}(\Upsilon)$, $\iota_{\Gamma_i, \Gamma}^$ the components of (43).*

(c) (Cutting an edge) *If Γ' is obtained from Γ by cutting an edge or edges with relations then*

$$\langle \alpha; \beta \rangle_{\Gamma, d, E} = \sum_k \langle \alpha, \delta_k \cup \text{Eul}_{G \times \mathbb{C}^\times}(E), \delta^k; \overline{\mathcal{M}}(\Upsilon)^* \beta \rangle_{\Gamma', d, E}$$

where $(\delta_k), (\delta^k), k = 1, \dots, \dim(H(I_{X//G}^m))$ are dual bases for $H(I_{X//G}^m)$ resp. $H_G(X)$ if the cut edges lie in $\text{Edge}^\infty(\Gamma)$ resp. $\text{Edge}^0(\Gamma)$.

(d) (Forgetting a tail) *If $\Upsilon : \Gamma \rightarrow \Gamma'$ is a morphism forgetting a tail then*

$$\langle \alpha, \alpha'; \overline{\mathcal{M}}(\Upsilon)^* \beta \rangle_{\Gamma, d, E} = (d, \alpha') \langle \alpha; \beta \rangle_{\Gamma', d, E}$$

where (d, α') is the pairing between $d \in H_2^G(X, \mathbb{Q})$ and $\alpha' \in H_G^2(X, \mathbb{Q})$.

Proof. By Proposition 8.2; the cutting edges case follows from an integration over the fiber $\overline{\mathcal{M}}_{n,\Gamma}^G(\mathbb{A}, X) \rightarrow \overline{\mathcal{M}}_{n,\Gamma_0}^G(\mathbb{A}, X)$ with fibers $\prod_{j>0} \overline{\mathcal{M}}_{0,n,\Gamma_j}(X)/G$. The collapsing edges and forgetting tails properties are left to the reader. \square

8.2. Quantum Kirwan morphism. In this section we use the affine gauged Gromov-Witten invariants to define the quantum Kirwan morphism from $QH_G(X)$ to $QH(X//G)$. For simplicity, we restrict to the case E trivial, that is, the untwisted case.

Definition 8.5. (Quantum Kirwan morphism) Suppose that X is a smooth polarized projective G -variety or a vector space with a linear action of G and proper moment map such that the git quotient $X//G$ is a Deligne-Mumford stack, so that the moduli stacks $\overline{\mathcal{M}}_n^G(\mathbb{A}, X)$ are proper Deligne-Mumford stacks. The *quantum Kirwan morphism* is the collection of maps

$$\kappa_X^{G,n} : QH_G(X)^n \times H(\overline{\mathcal{M}}_{n,1}(\mathbb{A})) \rightarrow QH(X//G), n \geq 0$$

given by pull-back to $\overline{\mathcal{M}}_{n,1}^G(\mathbb{A}, X)$ and push-forward to $X//G$, that is, for $\alpha \in H_G(X)^n, \alpha_\infty \in H_G(\overline{I}_{X//G}), \beta \in H^*(\overline{\mathcal{M}}_{n,1}(\mathbb{A}))$ let

$$(\kappa_X^{G,n}(\alpha, \beta), \alpha_\infty) = \sum_{d \in H_2^G(X, \mathbb{Q})} q^d \langle \alpha; \alpha_\infty; \beta \rangle_d$$

using Poincaré duality; the pairing on the left is given by cup product and integration over $\overline{I}_{X//G}$. Define $\kappa_G^0 \in H^*(\overline{\mathcal{M}}_{n,1}(\mathbb{A}))$ similarly, by integrating the unit.

Theorem 8.6. *The collection $\kappa_X^G = (\kappa_X^{G,n})_{n \geq 0}$ satisfies the axioms of a morphism of CohFT algebras.*

Proof. Note that $\kappa_X^{G,0}$ has contributions with coefficients q^d with $(d, \omega) > 0$, by the stability condition. It follows that the sum on the right-hand-side of (11) is finite modulo terms with coefficient q^a and higher, for any $a \in \mathbb{R}$. The equation (11) now follows from parts (a)-(c) of Proposition 8.4. \square

Remark 8.7. (a) (Equivariant quantum Kirwan morphism) If the action of G extends to an action of a group \tilde{G} containing G as a normal subgroup, there is a map

$$QH_{\tilde{G}}(X)^n \times H(\overline{\mathcal{M}}_{n,1}^G(\mathbb{A}, X)) \rightarrow QH_{\tilde{G}/G}(X//G)$$

defined by the same formula. After extending the coefficient ring of $QH_{\tilde{G}/G}(X//G)$ from $\Lambda_{X//G}$ to Λ_X^G we have a morphism of CohFT algebras

$$(44) \quad (\kappa_X^{\tilde{G}, G, n})_{n \geq 0} : QH_{\tilde{G}}(X) \rightarrow QH_{\tilde{G}/G}(X//G).$$

- (b) (Flatness of the quantum Kirwan morphism in the positive case) Suppose that $c_1^G(X)$ is semipositive in the sense that $(c_1^G(X), d) \geq 0$ for the homology class d of any gauged affine map. In this case, the “quantum corrections” in any $\kappa_X^{G,n}(\alpha_1, \dots, \alpha_n)$ are of degree at most $\deg(\alpha_1) + \dots + \deg(\alpha_n) + 2 - 2n$. In particular, the element $\kappa_X^{G,0}(1)$ can be written as the sum of elements of degree 0 and 2 with respect to the grading induced by the grading on $H(I_{X//G})$. If $c_1^G(X)$ is positive, then the dimension count shows that $\kappa_X^{G,0}$ is an element of degree 0 in $H(I_{X//G})$, times an element of Λ_X^G , that is, a multiple of the point class. If $(c_1^G(X), d)$ is at least two, whenever $(d, [\omega_{X,G}]) > 0$, then $\kappa_X^{G,0}$ vanishes.

We end this section with a partial computation of the quantum Kirwan morphism in the toric case. Suppose that $X \cong \mathbb{C}^k$ is a vector space equipped with a linear action of a torus G with Lie algebra \mathfrak{g} and weights $\mu_1, \dots, \mu_k \in \mathfrak{g}^\vee$ in the sense that G acts on the j -th factor by the character $\exp(\mu_j)$. We denote by $\tilde{G} = (\mathbb{C}^\times)^k$ the torus acting on X , and v_1, \dots, v_k the coordinates on the Lie algebra $\tilde{\mathfrak{g}}$ so that

$$QH_{\tilde{G}}(X) = \mathbb{Q}[v_1, \dots, v_k] \otimes \Lambda_X^{\tilde{G}}.$$

However, for the purposes of this section it suffices to tensor with the G -equivariant Novikov field Λ_X^G . The inclusion $G \rightarrow \tilde{G}$ induces a map $r : QH_{\tilde{G}}(X) \rightarrow QH_G(X)$, which after identification of the equivariant cohomology with symmetric functions $QH_G(X) \cong \text{Sym}(\mathfrak{g}^\vee) \otimes \Lambda_X^G$ is the restriction map induced by the inclusion $\mathfrak{g} \rightarrow \tilde{\mathfrak{g}}$. Let $l(v_j), j = 1, \dots, k$ denote the divisor classes in $H(I_{X//G})$ defined by v_j , see Example 4.8.

Lemma 8.8. *Let G be a torus acting on a vector space X as above. For any $d \in H_2^G(X, \mathbb{Z})$ such that $\nu \in \text{span}\{-\mu_j, \mu_j(d) \geq 0\}$ (see (29)) we have*

$$\kappa_X^{G,1} \left(\prod_{\mu_j(d) \geq 0} r(v_j)^{\mu_j(d)} \right) = q^d \prod_{\mu_j(d) \leq 0} l(v_j)^{-\mu_j(d)} + \text{higher order}$$

where higher order means terms with coefficient $q^{d'}$ with $(d', [\omega_{X,G}]) > (d, [\omega_{X,G}])$.

Proof. We show

$$(45) \quad \int_{[\overline{\mathcal{M}}_{1,1}^G(\mathbb{A}, X, d)]} \prod_{\mu_j(d) \geq 0} r(v_j)^{\mu_j(d)} \cup \text{ev}_\infty^* \alpha = \int_{[X//G]} \prod_{\mu_j(d) \leq 0} l(v_j)^{-\mu_j(d)} \cup \alpha.$$

We compute the left-hand-side by interpreting the first factor as an Euler class

$$\prod_{\mu_j(d) \geq 0} r(v_j)^{\mu_j(d)} = \text{ev}^* \text{Eul} \left(\bigoplus_{\mu_j(d) \geq 0} \mathbb{C}^{\mu_j(d)} \right)$$

and counting the zeros of a section. Identifying framed maps with a single marking with maps $u : \mathbb{A} \rightarrow X$, consider the map

$$\sigma : \overline{\mathcal{M}}_{1,1}^G(\mathbb{A}, X, d) \rightarrow \text{ev}_1^* \prod_{\mu_j(d) \geq 0} \mathbb{C}_{\mu_j}^{\mu_j(d)}, \quad u \mapsto (u_i^{(j)}(0))_{i=1, j=1}^{k, \mu_j(d)-1}$$

whose components are the derivatives of the map at the finite marking. On the stratum $\mathcal{M}_{1,1}^G(\mathbb{A}, X, d)$ of curves with irreducible domain, the intersection $\sigma^{-1}(0)$ maps injectively into $X//G \subset \overline{I}_{X//G}$ via ev_∞ . Indeed the assumption on the span of $\mu_j, \mu_j(d) \geq 0$ implies that $\mathcal{M}_{1,1}^G(\mathbb{A}, X, d)$ is non-empty, the equation $\text{ev}_\infty(u) = \text{pt}_{X//G}$ fixes the leading order terms (see Examples 5.32 and 8.9) and $\sigma(u) = 0$ fixes the lower order terms in u . Since ev_∞ maps smoothly onto $\bigoplus_{\mu_j(d) \geq 0} \mathbb{C}_{\mu_j} \cap X^{\text{ss}}/G$, the integral (45) is equal to

$$\int_{[X//G]} \alpha \cup \prod_{\mu_j(d) < 0} l(v_j)^{-\mu_j(d)}$$

where the virtual integration of $[u]$ is with respect to the virtual fundamental class induced from that on the moduli stack. Taking into account the *obstruction bundle*

$$R^1 p_* e^* T(X/G) = \text{ev}_1^* \bigoplus_{\mu_j(d) < 0} \mathbb{C}_{\mu_j}^{-\mu_j(d)-1}$$

we see that $\kappa_G^{X,1}(\prod r(v_j)^{\max(0, \mu_j(d))})$ contains a term of the form $q^d \prod l(v_j)^{\max(0, -\mu_j(d))}$ plus contributions from other strata and components of the moduli space of other homology classes.

We check next that there are no contributions from boundary strata. On the boundary with curves of reducible domain, each map u consists of component $u_1 : C_1 \rightarrow X/G$ consisting of an affine scaled map of homology class d' with $(d', [\omega_{X,G}]) < (d, [\omega_{X,G}])$ connecting the marking z_1 to the infinite marking z_0 , together with bubbles in $X//G$ and possibly other affine scaled maps. denote the map given by the derivatives of the curve at the marking. The vanishing $\sigma(u) = 0$ implies that, in particular, the $\mu_j(d')$ -th derivative of u_1 is zero if $\mu_j(d')$ is integral and less than some non-negative $\mu_j(d)$. The same conclusion holds if $\mu_j(d') < \mu_j(d)$ is negative,

since in this case the j -th component of u_1 vanishes identically. On the other hand, since $(d - d', [\omega_{X,G}]) > 0$, the set of points in $X//G$ whose j -th coordinate vanishes if $\mu_j(d - d') > 0$, is unstable, see (29). Thus, $\sigma^{-1}(0)$ is empty on the boundary strata and the only contribution to the integral above arises from the component of maps with irreducible domain. \square

Example 8.9. (a) (Projective Space Quotient) If $G = \mathbb{C}^\times$ acts on $X = \mathbb{C}^k$ with all weights one, so that $X//G = \mathbb{P}^{k-1}$, then $\mathcal{M}_{1,1}^G(\mathbb{A}, X)$ may be identified with the space of k -tuples of polynomials $(p_1(z), \dots, p_n(z))$ with $(p_1(z), \dots, p_k(z))$ semistable for z generic. We obtain a section

$$\sigma : \overline{\mathcal{M}}_{1,1}^G(\mathbb{A}, X, 1) \rightarrow \text{ev}_1^*(X \times X \rightarrow X)$$

by evaluating the polynomials and their derivatives at 0. One sees easily that this section has no zeroes other than at $(c_1 z, \dots, c_k z)$ for $(c_1, \dots, c_k) \neq 0$, which lies in the open stratum of maps with irreducible domain. In particular,

$$\int_{[\overline{\mathcal{M}}_{1,1}^G(\mathbb{A}, X, 1)]} \text{ev}_1^*(X \times X \rightarrow X) \cup \text{ev}_\infty^*([\text{pt}_{X//G}]) = 1$$

which implies that $\kappa_X^{G,1}(\xi^k) = q$ where ξ is the generator of $QH_G(X) \cong \Lambda_X^G[\xi]$.

- (b) (Weighted Projective Line Quotient) Let $X = \mathbb{C}_2 \oplus \mathbb{C}_3$ and $G = \mathbb{C}^\times$ so that $X//G = \mathbb{P}[2, 3]$. Let θ_1 resp. θ_2 resp. θ_3 denote the generator of the component of $QH(X//G) \cong H(\overline{T}_{X//G}) \otimes \Lambda_{X//G}$ with trivial isotropy resp. \mathbb{Z}_2 isotropy resp. corresponding to $\exp(\pm 2\pi i/3) \in \mathbb{Z}_3$. Let $\xi \in H_G^2(X)$ denote the integral generator. On each stratum of class j the section σ given by the derivatives of order less than j at the finite marking has a single, transverse zero in the locus of maps with irreducible domain, by the description in Example 5.32. On the other hand, σ has no zeroes in the boundary strata, since any boundary stratum has a component C_1 containing the finite marking on which u restricts to an affine gauged map with class $j' < j$, and vanishing of σ implies that the leading order terms vanish, so that $u|_{C_1}$ cannot be semistable at the node connecting C_1 with the root marking z_0 . Hence

$$\kappa_X^{G,1}(1) = 1, \kappa_X^{G,1}(\xi) = \theta_1, \quad \kappa_X^{G,1}(\xi^2) = q^{1/3}\theta_3/6,$$

$$\kappa_X^{G,1}(\xi^3) = q^{1/2}\theta_2/18, \kappa_X^{G,1}(\xi^4) = q^{2/3}\theta_3^2/36, \quad \kappa_X^{G,1}(\xi^5) = q/108.$$

In particular, we see that $\kappa_X^{G,1}$ is surjective and the kernel is $\xi^5 - q/108$, hence

$$QH(\mathbb{P}[2, 3]) = \mathbb{Q}[\xi] \otimes \Lambda_X^G / (\xi^5 - q/108)$$

which is a special case of Coates-Lee-Corti-Tseng [18].

Remark 8.10. (a) (Quantum Kirwan surjectivity) We conjecture the quantum analog of Kirwan surjectivity, namely that $\kappa_X^{G,1}$ is surjective onto the orbifold quantum cohomology $QH(X//G)$ of the quotient $X//G$. We have worked out some special cases with Gonzalez in [34], but the general case seems quite difficult.

- (b) (Quantum reduction in stages) One naturally expects a quantum analog of the reduction in stages theorem: If $G' \subset G$ is a normal subgroup then $\kappa_{X//G'}^{G/G'} \circ \kappa_X^{G,G'} = \kappa_X^G : QH_G(X) \rightarrow$

$QH(X//G)$ in the sense of (14). That is, we have a commutative diagram of CohFT algebras

$$\begin{array}{ccc} QH_G(X) & \xrightarrow{\kappa_X^G} & QH(X//G) \\ & \searrow \kappa_X^{G,G'} & \nearrow \kappa_{X/G'}^{G/G'} \\ & QH_{G/G'}(X//G') & \end{array}.$$

8.3. The adiabatic limit theorem. We show the adiabatic limit Theorem 1.5, using a divisor class relation on the moduli stack of stable scaled curves $\overline{\mathcal{M}}_{n,1}(C)$ relating curves with finite and infinite scaling. Note that divisor class relations in one-dimensional source moduli spaces have already been used to prove the associativity of the quantum products, as well as the homomorphism property of the quantum Kirwan morphism.

Recall from Theorem 5.35 the stack $\overline{\mathcal{M}}_{n,1}^G(C, X)$ of scaled gauged maps from C to X . Under the stable=semistable assumption it has a perfect relative obstruction theorem over $\overline{\mathfrak{M}}_{n,1}^{\text{tw}}(C)$, whose complex is dual to $Rp_*u^*T(X/G)$.

Definition 8.11. If every polystable gauged map is stable then the *scaled gauged Gromov-Witten invariants* for $\alpha \in H_G(X)^n, \beta \in H(\overline{\mathcal{M}}_{n,1}(C))$

$$\langle \alpha, \beta \rangle_{d,1,E} = \int_{[\overline{\mathcal{M}}_{n,1}^G(C,X,d)]} \text{ev}^* \alpha \cup f^* \beta \cup \epsilon(E).$$

Define

$$\phi^n : QH_G(X)^n \times H(\overline{\mathcal{M}}_{n,1}(C)) \rightarrow \Lambda_X^G, \quad (\alpha, \beta) \mapsto \sum_{d \in H_2^G(X, \mathbb{Q})} q^d \langle \alpha, \beta \rangle_{d,1,E}$$

for $\alpha \in H_G(X)^n$, extended to $QH_G(X)^n$ by linearity.

More generally there are invariants for arbitrary combinatorial type that satisfy the splitting axioms as in 7.15, 7.18 whose proof is similar. It follows:

Theorem 8.12. *The invariants $(\phi^n)_{n \geq 0}$ define a 2-morphism from the composition $\tau_{X//G,E//G} \circ \kappa_{X,E}^G$ to $\tau_{X,E}^G$ in the sense of 2.48.*

The adiabatic limit theorem Theorem 1.3 follows from Theorem 8.12, in particular from the divisor class relation

$$(46) \quad [\overline{\mathcal{M}}_n(C)] = [\cup_{r,[I_1,\dots,I_r]} \overline{\mathcal{M}}_r(C) \times \prod_{j=1}^r \overline{\mathcal{M}}_{|I_j|,1}(\mathbb{A})] \in H(\overline{\mathcal{M}}_{n,1}(C))$$

from Proposition 2.47.

9. LOCALIZED GRAPH POTENTIALS

In this section we make contact with the hypergeometric functions appearing in the work of Givental [31], Lian-Liu-Yau [55], Iritani [39] and others which compute the graph potential of the quotient, by studying the contributions to the localization formula for the circle action on the moduli spaces of gauged maps on the projective line. Note that in contrast to [31], [55] etc., the target can be an arbitrary projective (or in some cases, quasiprojective) G -variety.

The basic idea is to examine the action of \mathbb{C}^\times on the moduli stacks of gauged maps from \mathbb{P} to X induced by the standard action on \mathbb{P} . The virtual localization formula expresses the result as a sum over fixed point contributions, and comparing the contributions to the adiabatic limit Theorem 1.3 one obtains a stronger result which generalizes the “mirror theorems” of [31], [55], [39].

9.1. Liouville insertions. First we introduce a “Liouville class” in the definition of the graph potential, which helps to separate out the contributions from the various fixed points. Similar insertions appear in many places, for example, Witten’s treatment of two-dimensional gauge theory [96]. We first consider the case of ordinary Gromov-Witten theory with target X . Denote the universal curve and evaluation map

$$\begin{array}{ccc} \overline{\mathcal{C}}_n(C, X) & \xrightarrow{e \times e_C} & X \times C \\ p \downarrow & & \\ \overline{\mathcal{M}}_n(C, X) & & \end{array} .$$

Definition 9.1. (Liouville class and invariants with Liouville insertions) Let $\gamma \in H_G^2(X)$. The *Liouville class* associated to γ is

$$\lambda(\gamma) = \exp(p_*(e^* \gamma \cup e_C^*[\omega_C])) \in H_G(\overline{\mathcal{M}}_n(C, X)).$$

Replacing the virtual integrals in the definition of the graph invariants with virtual integrals

$$\int_{[\overline{\mathcal{M}}_n(C, X)]} \text{ev}^* \omega \cup f^* \beta \cup \epsilon(E) \cup \lambda(\gamma)$$

gives rise to graph invariants “with Liouville insertions”

$$(47) \quad H(X)^n \times H(\overline{\mathcal{M}}_n(C)) \otimes H^2(X) \rightarrow \mathbb{Q}[\zeta, \zeta^{-1}],$$

$$\langle \alpha, \beta, \gamma \rangle_{E, d} = \int_{[\overline{\mathcal{M}}_n(C, X, d)]} \text{ev}^* \alpha \cup f^* \beta \cup \lambda(\gamma) \cup \epsilon(E)$$

where ζ is the equivariant parameter for scalar multiplication on the fibers of $Rp_* u^* E$.

A similar definition of Liouville classes holds for stable gauged maps:

Definition 9.2. (Gauged Liouville class and invariants with Liouville insertions) Any class $\gamma \in H_G^2(X)$ gives rise to a *gauged Liouville class*

$$\lambda(\gamma) = \exp(p_*(e^* \gamma \cup e_C^*[\omega_C])) \in H(\overline{\mathcal{M}}_n^G(C, X)).$$

These give rise to invariants

$$(48) \quad H(X)^n \times H(\overline{\mathcal{M}}_n(C)) \otimes H^2(X) \rightarrow \mathbb{Q}[\zeta, \zeta^{-1}],$$

$$\langle \alpha, \beta, \gamma \rangle_{E, d} = \int_{[\overline{\mathcal{M}}_n^G(C, X, d)]} \text{ev}^* \alpha \cup f^* \beta \cup \lambda(\gamma) \cup \epsilon(E)$$

One could also take an odd class on C instead of $[\omega_C]$, restrict to maps of a fixed combinatorial type etc. but these possibilities will not be discussed here.

9.2. Localized equivariant graph potentials. In this section we discuss the extraction of a fundamental solution to the quantum differential equation from the graph potential, following e.g. Givental [31]. Let X be a smooth projective variety, or more generally, a smooth proper Deligne-Mumford stack with projective coarse moduli space. Let $C = \mathbb{P}$ be equipped with the standard \mathbb{C}^\times action with fixed points $0, \infty \in \mathbb{P}$. Denote by \hbar the equivariant parameter corresponding to the \mathbb{C}^\times -action. The notation is justified by the observation that the localization contributions given rise to a solution to the quantum differential equation (4). However note that the formal parameter q is also a “quantum parameter”. The graph potential τ_X has a natural \mathbb{C}^\times -equivariant generalization

$$\tau_X^{\mathbb{C}^\times} : QH(X) \times H_{\mathbb{C}^\times}(\overline{\mathcal{M}}_n(\mathbb{P})) \times H^2(X) \rightarrow \Lambda_X[[\hbar]].$$

For simplicity we restrict to the untwisted case, that is, E trivial. Following Givental [31] the \mathbb{C}^\times -equivariant graph potential can be computed via localization and is related to the fundamental solution of the quantum differential equation. The class $[\omega_{\mathbb{P}}] \in H^2(\mathbb{P})$ has a unique equivariant extension $[\omega_{\mathbb{P}, \mathbb{C}^\times}] \in H_{\mathbb{C}^\times}^2(\mathbb{P})$ taking values 0 resp. \hbar in $H_{\mathbb{C}^\times}(pt) \cong \mathbb{Q}[\hbar]$ after restriction to 0 resp. ∞ in \mathbb{P} . The following is well-known, see for example Givental [31].

Proposition 9.3. (Fixed points for the \mathbb{C}^\times -action on graph spaces) *The induced action of \mathbb{C}^\times on $\overline{\mathcal{M}}_n(\mathbb{P}, X, d)$ has fixed points given by configurations u consisting of a principal component $C_0 \cong \mathbb{P}$ on which the map u is constant, an n_- -marked stable map $u_- : C_- \rightarrow X$ of degree d_- attached to $0 \in \mathbb{P}$, and an n_+ -marked stable map $u_+ : C_+ \rightarrow X$ of degree d_+ attached to $\infty \in \mathbb{P}$ with $d_- + d_+ = d$ and $n_- + n_+ = n$.*

Lemma 9.4. (Restriction of Liouville class to fixed points) *The restriction of $\lambda(\gamma)$ to a component of the fixed point set consisting of a stable $n_- + 1$ -marked map of degree d_- attached to $0 \in \mathbb{P}$, a constant map from \mathbb{P} to X , and a stable $n_+ + 1$ -marked map of degree d_+ , is equal to $\exp((d_+, \gamma))$.*

Proof. Since the restriction of $p_*(e^*\gamma \cup e_C^*[\omega_C])$ to such a component is (γ, d_+) . \square

Definition 9.5. (Localized graph potentials) Define the *localized graph potentials* as the fixed point contribution to $\tau_{\mathbb{P}, \mathbb{C}^\times}(\alpha, \gamma)$ from 0 resp. ∞ from the components at 0 resp. ∞

$$\tau_{X, \pm} : QH(X) \rightarrow QH(X)[[\hbar^{-1}]]$$

by push-pull over the fixed point component given by $\overline{\mathcal{M}}_{0, n+1}(X, d)$ attached to a constant map $0 \in \mathbb{P}$:

$$\tau_{X, \pm}(\alpha, q, \hbar) := \sum_{n \geq 0} (1/n!) \tau_{X, \pm}^n(\alpha, \dots, \alpha, q, \hbar)$$

where

$$\tau_{X, \pm}^n(\alpha_1, \dots, \alpha_n, q, \hbar) = \sum_{d \in H_2^G(X, \mathbb{Z})} q^d \text{ev}_{n+1, *}(\mp \hbar(\pm \hbar - \psi_{n+1})^{-1} \bigcup_{i=1}^n \text{ev}_i^* \alpha_i)$$

and $\psi_{n+1} \in H^2(\overline{\mathcal{M}}_{0, n+1}(X, d))$ is the cotangent line at the $n + 1$ -st marked point.

Lemma 9.6. (Properties of localized graph potentials) *For $\alpha \in H(X)$, $\gamma \in H_G^2(X)$,*

- (a) (Duality) $\tau_{X, +}(\alpha, q, \hbar) = \tau_{X, -}(\alpha, q, -\hbar)$.
- (b) (Pairing) $\tau_X^{\mathbb{C}^\times}(\alpha, \gamma, q, \hbar) = \int_{[X]} \tau_{X, -}(\alpha, q, \hbar) \cup \tau_{X, +}(\alpha, qe^\gamma, \hbar) \cup \exp(\gamma)$.

Proof. The pairing formula follows from the virtual localization formula [35] for the \mathbb{C}^\times -action on $\overline{\mathcal{M}}_{0,n}(\mathbb{P}, X)$, using the description of the fixed points in Proposition 9.3 and the formula for the restriction of $\lambda(\gamma)$ in Lemma 9.4 which produces the shift in q . \square

Proposition 9.7. *The components of $\tau_{X,\pm}$ give solutions to the quantum differential equation (4) for the Frobenius manifold associated to the Gromov-Witten theory of X .*

This is a consequence of the topological recursion relations, see Pandharipande [78].

9.3. Localized gauged graph potentials. In this section we define a gauged version of the localized gauged potential. First we introduce the equivariant version of the gauged graph potential.

Definition 9.8. (Liouville class and invariants with Liouville insertions) Any class $\gamma \in H_G^2(X)$ gives rise to an equivariant class

$$\lambda(\gamma) = \exp(p_*(e^*\gamma \cup e_C^*[\omega_{C,\mathbb{C}^\times}])) \in H_{\mathbb{C}^\times}(\overline{\mathcal{M}}_n^G(C, X)).$$

Inserting this class in the integrals gives rise to gauged trace maps with Liouville insertions

$$\begin{aligned} \tau_X^{G,\mathbb{C}^\times,n} : H_G(X)^n \times H_{\mathbb{C}^\times}(\overline{\mathcal{M}}_n(C)) \times H_G^2(X) &\rightarrow \mathbb{Q}[[\hbar]] \\ (\alpha, \beta, \gamma) &\rightarrow \sum_{d \in H_2^G(X, \mathbb{Z})} q^d \int_{[\overline{\mathcal{M}}_n^G(C, X, d)_{\mathbb{C}^\times \rightarrow B\mathbb{C}^\times}]} \text{ev}^* \alpha \cup f^* \beta \cup \lambda(\gamma). \end{aligned}$$

The resulting potential, as in Givental [31], admits a “factorization” in terms of contributions to the fixed point formula near 0 and ∞ in \mathbb{P} ; the statement and proof take the remainder of this subsection.

The fixed point locus of the action \mathbb{C}^\times on $\overline{\mathcal{M}}_n^G(\mathbb{P}, X, d)$ is described in the lemma below.

Definition 9.9 (Clutching construction for gauged maps from \mathbb{P}). Given one-parameter subgroups $\phi_\pm : \mathbb{C}^\times \rightarrow G$ and an element $x \in X$, let $(\mathbb{C} \times G, u_\pm : \mathbb{C} \rightarrow X)$ denote the gauged maps over \mathbb{C} given by $u_\pm(z) = \phi_\pm(z)x$. Let $P(\phi_+, \phi_-)$ denote the bundle with clutching function $\phi_+(z)\phi_-(z^{-1})^{-1}$,

$$\begin{aligned} P(\phi_+, \phi_-) &= (\mathbb{C} \cup G) \cup_{\phi_+\phi_-^{-1}} (\mathbb{C} \cup G) \\ (r_\pm^* u(\phi_+, \phi_-, x))(z) &= \phi_\pm(z)x \end{aligned}$$

where r_\pm is restriction to the open subsets \mathbb{C} near 0 resp. ∞ .

Lemma 9.10. (Every fixed point arises from clutching) *Any \mathbb{C}^\times -fixed element $[P, u] \in \mathcal{M}^G(\mathbb{P}, X)^{\mathbb{C}^\times}$ is of the form $P = P(\phi_+, \phi_-)$, $u = u(\phi_+, \phi_-, x)$ for some ϕ_+, ϕ_-, x as in Definition 9.9.*

Proof. The section u is fixed up to an automorphism of P which must be given in trivializations near $0, \infty$ by characters $\phi_\pm : \mathbb{C}^\times \rightarrow G$. Since the section is global, the clutching map is $\phi_+\phi_-^{-1}$. \square

To investigate the stability of a map formed by the clutching construction in Lemma 9.10, we restrict to the case that G is a torus with Lie algebra \mathfrak{g} and weight lattice $\Lambda^\vee \subset \mathfrak{g}^\vee$.

Definition 9.11. Identify the first Chern class $c_1(P) \in H^2(C, \Lambda^\vee)$ with an element of Λ^\vee via tensor product with the fundamental class of C .

- (a) (Shifted polarization) $c_1(P)$ defines a *shifted polarization* on X , given by tensoring the given polarization $L \rightarrow X$ with the rational character given by $\rho^{-1}c_1(P)$. In terms of the moment map description, the corresponding quotient is the inverse image of $\rho^{-1}c_1(P)$, modulo the action of G .
- (b) (Shifted semistability) We say that a point $x \in X$ is *shifted semistable* if x is semistable with respect to the shifted polarization.
- (c) (Shifted git quotient) Denote by $X//_{\rho^{-1}c_1(P)}G$ the git quotient with respect to the shifted polarization.

Lemma 9.12. (Semistability of gauged maps formed by clutching) *Suppose that $\phi_{\pm} : \mathbb{C}^{\times} \rightarrow G$ and x is a point such that $\lim_{z \rightarrow 0} \phi_{\pm}(z)x$ exists. Then the pair $(P = P(\phi_{-}, \phi_{+}), u = u(\phi_{-}, x, \phi_{+}))$ given by the clutching construction is Mundet semistable iff x is shifted semistable.*

Proof. With (P, u) as in the statement of the Lemma, the limit of $u(z)$ with respect to any one-parameter subgroup in the maximal torus containing λ satisfies $\int_{[C]} (\text{Gr}(u) \circ \Phi[\omega_C] + \rho^{-1}c_1(P), \lambda) \leq 0$, iff $u(z)$ is shifted semistable for $z \in \mathbb{C}^{\times}$. Since $u(1) = x$, the lemma follows. \square

Corollary 9.13. (Clutching description of the circle-fixed gauged maps) *Each component of $\mathcal{M}_2^G(\mathbb{P}, X, d)^{\mathbb{C}^{\times}}$ is isomorphic to $X//_{\rho^{-1}(c_1(P))}G$ with evaluation maps given by $\lim_{z \rightarrow 0} \phi_{\pm}(z)x$ for some one-parameter subgroups $\phi_{\pm} : \mathbb{C}^{\times} \rightarrow X$. Furthermore, each component of $\overline{\mathcal{M}}_n^G(\mathbb{P}, X, d)^{\mathbb{C}^{\times}}$ is isomorphic to a fiber product*

$$(\overline{\mathcal{M}}_{0, n_{-}+1}(X, d_{-}) \times_X \mathcal{M}_2^{G, \text{fr}}(\mathbb{P}, X, d_0)^{\mathbb{C}^{\times}} \times_X \overline{\mathcal{M}}_{0, n_{+}+1}(X, d_{+}))/G^2$$

for some $d_{-} + d_0 + d_{+} = d$ and $n_{-} + n_{+} = n$.

Proof. The fixed points corresponding to data (ϕ_{+}, ϕ_{-}, x) such that the corresponding sections $\phi_{\pm}(z)x$ extend over 0. The second decomposition describes the bubble trees attached to 0, ∞ . \square

Example 9.14. (Projective space quotient) Let $X = \mathbb{C}^k$ with $G = \mathbb{C}^{\times}$ acting diagonally. There are no holomorphic curves in X , hence d_{\pm} always vanish. The moduli stack of gauged maps of class $d \in H_2^G(X, \mathbb{Z}) \cong \mathbb{Z}$ is \mathbb{P}^{kd+k-1} , the projective space of k -tuples of polynomials in two variables of degree d . The group \mathbb{C}^{\times} by pull-back, with fixed point set $\mathcal{M}(\mathbb{P}, X, d)^{\mathbb{C}^{\times}}$ the union of projective spaces of k -tuples of homogeneous polynomials of some degree $i = 0, \dots, d$, each isomorphic to \mathbb{P}^{k-1} . Thus $\phi_{-} = i, \phi_{+} = d - i$, and the isomorphism is given by evaluation at a generic point.

Lemma 9.15. (Restriction of gauged Liouville classes to fixed point components) *The restriction of $\lambda(\gamma)$ to a fixed point component of $\overline{\mathcal{M}}_n^G(\mathbb{P}, X)$ consisting of stable $n_{-} + 1$ -marked maps of degree d_{-} attached to $0 \in \mathbb{P}$, maps from \mathbb{P} to X/G given by data (ϕ_{-}, x, ϕ_{+}) , and stable $n_{+} + 1$ -marked maps of degree d_{+} , is equal to $\exp((d_{+} + \phi_{+}, \gamma)\hbar)$.*

Proof. Since the restriction of $p_*(e^*\gamma \cup e_C^*[\omega_C])$ to such a configuration is $(\gamma, d_{+} + \phi_{+})\hbar$: the term (γ, d_{+}) arises from the integration over the bubble attached to ∞ while the second arises from integration over the principal component, which may be computed via localization for the \mathbb{C}^{\times} -action which has a single fixed point with restriction of $[\omega_C]$ given by \hbar and restriction of $e^*\gamma$ given by $(\gamma, \phi_{+})\hbar$. \square

We can describe the fixed point set of the \mathbb{C}^\times -action in terms of the data ϕ_\pm and bubble trees attached to the fixed points alternatively as follows.

Definition 9.16. (Localized stack of gauged maps) Let $\overline{\mathcal{M}}_n^G(\mathbb{P}, X, d_\pm)_{\pm}^{\mathbb{C}^\times}$ be the stack of pairs (P, u) consisting of a principal G -bundle on P on the principal component given in terms of trivializations $P|_{\mathbb{C}_\pm} \cong \mathbb{C}_\pm \times G$ by a transition map given by a one-parameter subgroup $\phi_\pm : \mathbb{C}^\times \rightarrow G$, a section u constant in the trivialization over \mathbb{C}_\mp , and all markings and bubbles over 0 resp. ∞ . Denote by

$$(49) \quad \text{ev}_{\pm, \infty} : \overline{\mathcal{M}}_n^G(\mathbb{P}, X, d_\pm)_{\pm}^{\mathbb{C}^\times} \rightarrow X/G$$

the evaluation map at $0 \in \mathbb{C}_\mp$. Since u is constant in this trivialization, the evaluation map takes values in the shifted git quotient $X //_{\rho^{-1}d} G$.

Remark 9.17. (a) (Fixed point sets as fiber products) From the description of the fixed point sets above we see that

$$(50) \quad \overline{\mathcal{M}}_n^G(\mathbb{P}, X, d)_{\pm}^{\mathbb{C}^\times} = \bigcup_{d_- + d_+ = d} \bigcup_{n_- + n_+ = n} \overline{\mathcal{M}}_{n_-}^G(\mathbb{P}, X, d_-)_{-}^{\mathbb{C}^\times} \times_{(X //_{\rho^{-1}(d)} G)} \overline{\mathcal{M}}_{n_+}^G(\mathbb{P}, X, d_+)_{+}^{\mathbb{C}^\times}$$

where the fiber products use the evaluation maps (49).

(b) (Localized stacks as fixed point components) By taking $\phi_\mp = 0$ one sees that $\overline{\mathcal{M}}_{n_\pm}^G(\mathbb{P}, X, d_\pm)_{\pm}^{\mathbb{C}^\times}$ is a component of $\overline{\mathcal{M}}_{n_\pm}^G(\mathbb{P}, X, d_\pm)_{\pm}^{\mathbb{C}^\times}$ and so proper and equipped with virtual fundamental classes.

Definition 9.18. (a) (Normal complexes) Let $N_\pm(n_\pm, d_\pm)$ denote normal complex of the embedding of $\overline{\mathcal{M}}_{n_\pm}^G(\mathbb{P}, X, d_\pm)_{\pm}^{\mathbb{C}^\times}$ in $\overline{\mathcal{M}}_{n_\pm}^G(\mathbb{P}, X, d_\pm)$. Splitting the normal complex into the contributions from deformations of the map, deformations of the node at the principal component, and deformations of the attaching point to the principal component we have

$$N_\pm(n_\pm, d_\pm) = (Rp_* u^*(T(X/G)))^+ \oplus T_{w_+}^\vee C \otimes T_{w_-}^\vee \hat{C}^\rho \oplus T_{w_+} C$$

where $(Rp_* u^*(T(X/G)))^+$ is the moving part (under the action of \mathbb{C}^\times) of the index of the tangent complex of X/G and $w_\pm \in \hat{C}^\rho$ are the preimages of the node connecting to the principal component at 0 in the normalization \hat{C}^ρ , so that $w_+ = 0$ in the principal component identified with C .

(b) (Euler class of the normal complex) The Euler class of $N_\pm(n_\pm, d_\pm)$ is, assuming the existence of a bubble attached at $0 \in C$,

$$\text{Eul}((Rp_* u^*(T(X/G)))^+)(\pm \hbar)(\psi \mp \hbar)$$

where ψ is the cotangent line of the node of the bubble component at 0.

(c) (Localized Gauged Potentials) The *localized gauged potentials* $\tau_{X, \pm}^G$ are the contributions from the configurations with $d_\mp = 0$:

$$(51) \quad \tau_{X, \pm}^G : QH_G(X) \rightarrow QH_G(X)[[\hbar^{-1}]], \quad \tau_{X, \pm}^G(\alpha, q, \hbar) = \sum_{n \geq 0} (1/n!) \tau_{X, \pm}^{G, n}(\alpha, \dots, \alpha, q, \hbar)$$

$$\tau_{X, \pm}^{G, n}(\alpha_1, \dots, \alpha_n, q, \hbar) = \sum_d q^d \text{ev}_{\infty, *}(\text{ev}_1^* \alpha_1 \cup \dots \cup \text{ev}_n^* \alpha_n \cup \text{Eul}(N_\pm(n_\pm, d_\pm)))^{-1}.$$

We now establish the relationship between the localized gauged potentials and the gauged graph potential.

Definition 9.19. (Shifted pairing) Define on $QH_G(X)$ a *shifted pairing*

$$(\ , \)_\rho : QH_G(X) \times QH_G(X) \rightarrow \mathbb{Q}$$

given by

$$q^d \alpha \mapsto \int_{[X//_{\rho^{-1}\pi(d)}G]} \kappa_{X,\rho^{-1}\pi(d)}^{G,0}(\alpha) \in \mathbb{Q}$$

where $\pi(d) \in H_2(BG)$ is the image of $d \in H_2^G(X)$ (that is, the first Chern class of the bundle) and

$$\kappa_{X,\rho^{-1}\pi(d)}^{G,0} : H_G(X) \rightarrow H(X//_{\rho^{-1}\pi(d)}G)$$

is the classical Kirwan map for the git quotient $X//_{\rho^{-1}\pi(d)}G$, which we assume is locally free for all d .

Proposition 9.20. (Properties of localized gauged potentials)

- (a) (Duality) $\tau_{X,+}^G(q, \alpha, \hbar) = \tau_{X,-}^G(q, \alpha, -\hbar)$.
- (b) (Pairing) $\tau_{X,\mathbb{C}^\times}^G(\alpha, \gamma, q, \hbar) = (\tau_{X,-}^G(\alpha, q, \hbar), \tau_{X,+}^G(\alpha, qe^\gamma, \hbar)_\rho)$.

Here the pull-back action $\exp(\hbar\gamma) \in H_2^G(X, \mathbb{Z})$ on $\Lambda_X^G[[\hbar]]$ is

$$(f(q) = \sum c_d q^d) \mapsto (f(q \exp(\hbar\gamma)) = \sum c_d q^d \exp(\hbar(\gamma, d))).$$

Proof. (a) The automorphism $z \mapsto 1/z$ relates the stacks $\overline{\mathcal{M}}_n(\mathbb{P}, X, d)_\pm^{\mathbb{C}^\times}$, inverting the circle action and changing the class $\exp(p_* e^* \gamma \cup e_C^* \omega_{C, \mathbb{C}^\times})$ by $\exp((\gamma, d)\hbar)$. (b) is a consequence of localization and (50). \square

Example 9.21. (Localized gauged graph potential for toric quotients) Let G be a torus acting on a vector space X is a vector space with weights μ_1, \dots, μ_k . Expanding the definitions of the Euler classes we have

$$\tau_{X,-}^G(1, \hbar, q) = \sum_{d \in H_2^G(X)} q^d \frac{\prod_{j=1}^k \prod_{m=-\infty}^{\mu_j(d)} (\mu_j + m\hbar)}{\prod_{j=1}^k \prod_{m=-\infty}^0 (\mu_j + m\hbar)}.$$

The function $\tau_{X,-}^G(\alpha, \hbar, q)$ for $\alpha \in H_G(X)$ can be computed as follows. Since there are no non-constant holomorphic spheres in X , the evaluation maps $\text{ev}_1, \dots, \text{ev}_n$ are equal on $\overline{\mathcal{M}}_n^G(\mathbb{P}, X)_-$. It follows that the pushforward of $\text{ev}^* \alpha^{\otimes n} / (\hbar - \psi)$ under $\text{ev}_{\infty,-}$ is equal to

$$(52) \quad \alpha^n(\hbar)^{-2} \int_{[\overline{\mathcal{M}}_{0,n+1}]} (\psi_{n+1}/\hbar)^{n-2}.$$

This integral can be computed iteratively by pushing forward under the maps f_i forgetting the i -th marked point for $i \leq n$. We have the relation for the first Chern class ψ_{n+1} of the cotangent line at the last marked point in $\overline{\mathcal{M}}_{0,n+1}$

$$\psi_{n+1} = f_i^* \psi_n + [D_{\{0,i,n+1\} \cup \{1,\dots,\hat{i},\dots,n\}}] \in H(\overline{\mathcal{M}}_{0,n+1}).$$

The divisor class is degree one in any fiber of the forgetful map f_i and it follows that the integral (52) is equal to $(\alpha/\hbar)^n$. This implies that for $\alpha \in H_G(X)$

$$\tau_{X,-}^G(\alpha, \hbar, q) = \exp(\alpha/\hbar) \tau_{X,-}^G(1, \hbar, q).$$

Thus $\tau_{X,-}^G$ is Givental's *I-function*, see [32]. A special feature of the case X affine (so no sphere bubbles) is that this relation holds for α of arbitrary degree, not just divisor classes.

9.4. Localized adiabatic limit theorem. We prove the refinement Theorem 1.6 of the adiabatic limit Theorem 1.5 by comparing the fixed point contributions to the graph potentials.

Define a \mathbb{C}^\times -equivariant extension of the quantum Kirwan map

$$\kappa_{X,G} : QH_G(X) \rightarrow QH(X//G)[[\hbar]]$$

by pushing-forward over $\overline{\mathcal{M}}_n^G(\mathbb{A}, X)$ \mathbb{C}^\times -equivariantly, as follows. Choose a base point in \mathbb{A} , inducing an identification $\mathbb{A} \rightarrow \mathbb{C}$ and so a \mathbb{C}^\times -action on \mathbb{A} . The action induces an action on $\mathcal{M}_n^G(\mathbb{A}, X)$, given by pre-composing each morphism with the action. Equivalently, the action is given by acting by scalar multiplication on the one-form λ , and so extends to the compactification $\overline{\mathcal{M}}_n^G(\mathbb{A}, X)$. As a result, the virtual fundamental class for $\overline{\mathcal{M}}_n^G(\mathbb{A}, X)$ has a \mathbb{C}^\times -equivariant extension which can be used to define \mathbb{C}^\times -equivariant extension of $\kappa_{X,G}$.

Lemma 9.22. (Existence of a circle action on the master space) *The \mathbb{C}^\times -action on $\overline{\mathcal{M}}_n^G(\mathbb{P}, X)$ extends to $\overline{\mathcal{M}}_{n,1}^G(\mathbb{P}, X)$ so that the action on the substack $\overline{\mathcal{M}}_n^G(\mathbb{P}, X)_\infty$ is the action induced from the action of \mathbb{C}^\times on the factors $\overline{\mathcal{M}}_{i_j,1}^G(\mathbb{A}, X)$ and $\overline{\mathcal{M}}_r^G(\mathbb{P}, X//G)$.*

Proof. It suffices to check that the \mathbb{C}^\times -action is induced from a \mathbb{C}^\times -action on the universal curve $\overline{\mathcal{C}}_{n,1}(\mathbb{P}) \rightarrow \overline{\mathcal{M}}_{n,1}(\mathbb{P})$. For this note that the \mathbb{C}^\times -action on \mathbb{P} induces an action on stable maps to \mathbb{P} , by composition, and on the relative dualizing sheaf. Hence \mathbb{C}^\times acts on $\overline{\mathcal{C}}_{n,1}(\mathbb{P})$, and for the substack $\overline{\mathcal{M}}_n^G(\mathbb{P}, X)_\infty$ consisting of fiber products of $\overline{\mathcal{M}}_{i_j,1}^G(\mathbb{A}, X)$ and $\overline{\mathcal{M}}_r^G(\mathbb{P}, X//G)$, the action is given by rotation on \mathbb{P} and action on the one-forms in the objects of $\overline{\mathcal{M}}_{i_j,1}^G(\mathbb{A}, X)$, as claimed. \square

The proof of the adiabatic limit theorem goes through \mathbb{C}^\times -equivariantly, with Liouville insertions, by the same argument. In particular, the class $\lambda(\gamma)$ extends to a \mathbb{C}^\times -equivariant class on $\overline{\mathcal{M}}_{n,1}^G(\mathbb{P}, X, d)$. From the divisor class relation one obtains a \mathbb{C}^\times -equivariant extension of the equality

$$(53) \quad \lim_{\rho \rightarrow \infty} \tau_X^{G, \mathbb{C}^\times, n}(\alpha, \beta) = \sum_{[I_1, \dots, I_r]} \tau_{X//G}^{\mathbb{C}^\times, r}(\kappa_X^{G, |I_1|}(\alpha_{I_1}, \cdot), \dots, \kappa_X^{G, |I_r|}(\alpha_{I_r}, \cdot), \cdot)(\beta).$$

In the proof below we compare the fixed point contributions from both sides.

Proof of Theorem 1.6. The equality in Theorem 1.6 follows from a comparison of the terms in the virtual localization for the \mathbb{C}^\times action in the adiabatic limit formula. More precisely, we wish to extract the “lowest order” part of the equality arising from the localization formula, using Liouville insertions.

Consider the equality (53). As a function in the parameter \hbar , both sides are finite linear combinations of polynomials times exponentials. By Fourier transform, the contributions with the same exponents must match for any α, β . We choose

$$\beta \in H^{\mathbb{C}^\times}(\overline{\mathcal{M}}_{n+1}(\mathbb{P})), \quad \beta = \prod_{i=1}^n \text{ev}_{\mathbb{P},i}^* \exp([\omega_{\mathbb{P},\mathbb{C}^\times}]) \cup \text{ev}_{\mathbb{P},n+1}^* [\infty]$$

where $[\infty]$ is the dual class of the point $\infty \in \mathbb{P}$ and $\text{ev}_{\mathbb{P},i} : \overline{\mathcal{M}}_{n+1}(\mathbb{P}) \rightarrow \mathbb{P}$ is the i -th evaluation map. For $\gamma = [\omega_{X,G}] \in H_G^2(X)$ the Kähler class, the values of the equivariant part of $\lambda(\gamma)$ on the fixed point components of $\overline{\mathcal{M}}_{n+1}(\mathbb{P}, X//G)$ range from 1 when $d_- = d$ to $\exp((d, [\omega_{X,G}])\hbar)$ when $d_+ = d$, by Lemma 9.4. Similarly, the values of the equivariant part of $\lambda(\gamma)$ on the fixed point components of \mathbb{C}^\times on $\overline{\mathcal{M}}_n^G(\mathbb{P}, X)$ range from 1 when $d_- = d, \phi_+ = 0$ to $\exp((d, [\omega_{X,G}])\hbar)$ when $d_+ + \phi_+ = d$, by Lemma 9.15. Finally, the values of $\text{ev}_{\mathbb{P},i}^* \exp([\omega_{\mathbb{P},\mathbb{C}^\times}])$ range from 1 to $\exp(\hbar)$, if the i -th marked point maps to ∞ .

We compute the integral over $\overline{\mathcal{M}}_n^G(\mathbb{P}, X)_\infty$ by first pushing forward to $\overline{\mathcal{M}}_r(\mathbb{P}, X//G)$, and then applying the localization formula. The lowest order terms arise from bubble trees attached only at $0 \in \mathbb{P}$. In particular, the coefficient of the term with minimal exponent ($d_+ = \phi_+ = 0$ and the first n markings mapping to $0 \in \mathbb{P}$) must match the contribution from configurations with minimal exponent on the right-hand-side, consisting of configurations with the first n markings mapping to $0 \in \mathbb{P}$ and trivial homology class of the gauged map attaching to $0 \in \mathbb{P}$. This implies

$$(54) \quad \sum_{[I_1, \dots, I_r, \{n+1\}]} \int_{[X//G]} (\tau_{X//G,-}^{\mathbb{C}^\times, r}(\kappa_X^{G, |I_1|}(\alpha_{I_1}, 1), \dots, \kappa_X^{G, |I_r|}(\alpha_{I_r}, 1)) \cup \kappa_X^{G, \text{class}}(\alpha_{n+1})) \\ = \lim_{\rho \rightarrow \infty} \int_{[X//G]} (\kappa_X^{G, \text{class}} \circ \tau_{X,-}^{G, n})(\alpha_1, \dots, \alpha_n) \cup \kappa_X^{G, \text{class}}(\alpha_\infty).$$

Summing over n with $\alpha_1 = \dots \alpha_n = \alpha$ and $\alpha_{n+1} = \infty_\infty$ gives

$$\int_{[X//G]} (\tau_{X//G,-} \circ \kappa_X^G)(\alpha) \cup \kappa_X^{G, \text{class}}(\alpha_\infty) = \lim_{\rho \rightarrow \infty} \int_{[X//G]} (\kappa_X^{G, \text{class}} \circ \tau_{X,-}^G)(\alpha) \cup \kappa_X^{G, \text{class}}(\alpha_\infty)$$

hence Theorem 1.6. \square

The localized adiabatic limit Theorem 1.6 allows us to deduce relations in the quantum cohomology algebras, although these relations are rather non-explicit unless κ_X^G is known. Namely, recall that any differential operator annihilating the localized graph potential $\tau_{X//G,-}$ defines relations, see for example [31], [40] etc.

Corollary 9.23. (Relations on quantum cohomology algebras) *Suppose that \square is a differential operator on $QH_G(X)$ that admits a push-forward under $(\kappa_X^G)_*$ to a differential operator on $QH(X//G)$. (For example, \square could have coefficients in Λ_X^G .)*

(a) (Annihilation at a point) *If \square annihilates*

$$(55) \quad \mathfrak{J}_X^G : QH_G(X) \rightarrow QH(X//G), \quad \alpha \mapsto \kappa_X^{G, \text{class}}(\lim_{\rho \rightarrow \infty} \tau_{X,-}^G)$$

at $\alpha \in QH_G(X)$, then $(\kappa_X^G)_* \square$ annihilates $\tau_{X//G,-}$ at $\kappa_X^G(\alpha)$ and the symbol $\Sigma((\kappa_X^G)_* \square)$ of $(\kappa_X^G)_* \square$ satisfies $\Sigma((\kappa_X^G)_* \square) \star_{\kappa_X^G(\alpha)} \tau_{X//G,-}(\alpha) = 0$.

- (b) (Annihilation on big quantum cohomology) If $(\kappa_X^G)_* \square$ annihilates \mathfrak{I}_X^G at all $\alpha \in QH_G(X)$, then $\Sigma((\kappa_X^G)_* \square)(\alpha)$ is a relation in $T_{\kappa_X^G(\alpha)} QH(X//G)$ for all α .
- (c) (Annihilation on small quantum cohomology) If $(\kappa_X^G)_* \square$ annihilates \mathfrak{I}_X^G at all $\alpha \in QH_G^2(X)$, and $QH(X//G)$ is generated by the image of $QH_G^2(X)$, then $\Sigma((\kappa_X^G)_* \square)(\alpha)$ is a relation in $T_{\kappa_X^G(\alpha)} QH(X//G)$ for all $\alpha \in QH_G^2(X)$.

Proof. Suppose \square is a differential operator as in the Corollary part (a). That $(\kappa_X^G)_* \square$ annihilates $\tau_{X//G,-}$ follows from Theorem 1.6. The relation on the principal symbol follows from the fact that $\tau_{X//G,-}$ is a fundamental solution to the quantum differential equation (4), and so differential operators transform into quantum multiplications at $\kappa_X^G(\alpha)$. That the principal symbol defines relations follows as in [31], [40, Theorem 2.4], using that $\tau_{X//G,-}$ generates the quantum cohomology resp. derivatives of $\tau_{X//G,-}$ in the directions $(D\kappa_X^G)(\alpha)$, $\alpha \in QH_G^2(X)$ under the $QH_G^2(X)$ -generation hypothesis. \square

Remark 9.24. (a) (Alternative proof without the localized adiabatic limit theorem) An alternative proof of items (b), (c) is given as follows: because \square_d annihilates \mathcal{I}_X^G (resp. restricted to $QH_G^2(X)$), it annihilates the Hessian of the gauged graph potential τ_X^G , hence by the adiabatic limit Theorem 1.5, $(\kappa_X^G)_* \square_d$ annihilates the Hessian of the graph potential $\tau_{X//G}$ (resp. on the image of $QH_G^2(X)$). As explained in Givental [31], the Hessian gives a fundamental solution to the quantum differential equation and so the corresponding product vanishes in $QH_{\kappa_X^G(0)}(X//G)$ (resp. assuming generation by the image of $QH_G^2(X)$).

- (b) (The curvature is not necessarily small) The point $\kappa_X^G(0) \in QH(X//G)$ does not necessarily lie in $H_G^2(X) \otimes \Lambda_X^G$ unless $X//G$ is semipositive, so the above proposition gives relations not necessarily in the small quantum cohomology of $X//G$.

In the remainder of the section we discuss the toric case, that is, X is a complex vector space and G is a torus acting on X so that $X//G$ is a smooth proper Deligne-Mumford stack. In this case, The identity in Theorem 1.6 seems to be essentially the same as the “mirror theorems” in [31], [55], [39] etc. Regarding relations in the quantum cohomology of toric varieties, the following is introduced in Batyrev [7].

Definition 9.25. The *quantum Stanley-Reisner ideal* is the ideal $QSR_X^G \subset QH_G(X)$ generated by the elements

$$\prod_{\mu_j(d) > 0} r(v_j)^{\mu_j(d)} - q^d \prod_{\mu_j(d) < 0} r(v_j)^{\mu_j(d)} \in QH_G(X)$$

for $d \in H_2^G(X, \mathbb{Z}) \cong \mathfrak{g}_{\mathbb{Z}}$. Similarly the *equivariant quantum Stanley-Reisner ideal* is the ideal $Q\tilde{S}R_X^{\tilde{G}, G} \subset QH_{\tilde{G}}(X)$ generated by the elements

$$\prod_{\mu_j(d) > 0} v_j^{\mu_j(d)} - q^d \prod_{\mu_j(d) < 0} v_j^{\mu_j(d)} \in QH_{\tilde{G}}(X),$$

that is, without restriction to $\mathfrak{g} \subset \tilde{\mathfrak{g}}$.

Remark 9.26. (Why not primitive classes?) Batyrev [7] restricts to *primitive* classes d corresponding to collections $I \subset \{1, \dots, k\}$ such that I is a minimal subset such that the product of divisors $\prod_{i \in I} l(v_i)$ is zero in $H(X//G)$. The approach here seems better suited to the orbifold case. It would be interesting to find minimal generators for the quantum cohomology of toric orbifolds.

Example 9.27. (a) (Projective spaces) For the usual action of $G = \mathbb{C}^\times$ on $X = \mathbb{C}^k$, we have $H_G^2(X, \mathbb{Z}) = \mathbb{Z}$ with positive generator $d = 1$ and $r(v_j) = u$ the coordinate on \mathfrak{g} for $j = 1, \dots, k$. The quantum Stanley-Reisner ideal QSR_X^G has generator

$$\prod_{j=1}^k r(v_j) - q = u^k - q$$

while the equivariant Stanley-Reisner ideal $QSR_X^{\tilde{G}, G}$ has generator

$$\prod_{j=1}^k r(v_j) - q = v_1 \dots v_k - q.$$

(b) (Weighted projective line) Suppose that $G = \mathbb{C}^\times$ acts on $X = \mathbb{C}^2$ with weights $\mu_1 = 2, \mu_2 = 3$. Then $H_2^G(X) \cong \mathbb{Z}$ with generator $d = 1$. We have $r(v_1) = 2u$ while $r(v_2) = 3u$. The quantum Stanley-Reisner ideal QSR_X^G has generator

$$\prod_{j=1}^2 r(v_j)^{\mu_j(1)} - q = (2u)^2(3u)^3 - q$$

while the equivariant Stanley-Reisner ideal $QSR_X^{\tilde{G}, G}$ has generator

$$\prod_{j=1}^2 v_j^{\mu_j(1)} - q = v_1^2 v_2^3 - q.$$

Recall the definition of $\kappa_X^{\tilde{G}, G}$ from Remark 8.7.

Theorem 9.28. *The kernel of $D_\alpha \kappa_X^{G, 1} : T_\alpha QH_G(X) \rightarrow T_{\kappa_X^G(\alpha)} QH(X//G)$ resp. $D_\alpha \kappa_X^{\tilde{G}, G} : T_\alpha QH_G(X) \rightarrow T_{\kappa_X^{\tilde{G}, G}(\alpha)} QH_T(X//G)$ contains QSR_X^G resp. $QSR_X^{\tilde{G}, G}$.*

Proof. For $d \in H_2^G(X, \mathbb{Z}) \cong H(BG, \mathbb{Z}) \cong \mathfrak{g}_\mathbb{Z}^\vee$ let \square_d denote the differential operator on $QH_2^G(X, \mathbb{R})$ corresponding to d ,

$$\square_d = \prod_{\mu_j(d) \geq 0} \partial_j^{\mu_j(d)} - q^d \prod_{\mu_j(d) \leq 0} \partial_j^{-\mu_j(d)}.$$

We may identify the coordinates on $QH_2^G(X, \mathbb{R})$ with the quantum parameters, using the divisor equation. Then the operator \square_d annihilates the function of Example 9.21, see for example Iritani [39], Cox-Katz [19, (11.92)]. It follows from Corollary 9.23 that the corresponding product of the tangent vectors maps to zero in $T_{\kappa_X^G(\alpha)} QH(X//G)$, and so lies in the kernel of $D_\alpha \kappa_X^G$. \square

Remark 9.29. (Isomorphism with the quantum Stanley-Reisner ring) In joint work with Gonzalez [34], we show that $\kappa_X^{G,1}$ is surjective and QSR_X^G is exactly its kernel, after passing to a suitable formal version of $QH_G(X)$, so that $T_{\kappa_X^G(0)}QH(X//G)$ is canonically isomorphic to the quantum Stanley-Reisner ring. Related computations can be found in McDuff-Tolman [63] and Iritani [41].

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MATHEMATICS-HILL CENTER, RUTGERS UNIVERSITY, 110 FRELINGHUYSEN ROAD, PISCATAWAY, NJ 08854-8019, U.S.A.

E-mail address: `ctw@math.rutgers.edu`